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Characterizations of Multiplicative Linear Functionals on Banach Algebras

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Abstract
Let A be a Banach algebra. It is well known that there exist non-zero algebra homomorphisms from A into the complex field $\mathbb{C}$, equivalently, there exist non-zero multiplicative linear functionals defined on A. Such functionals play a central role in the structural analysis of Banach algebras, as they are closely related to maximal ideals. In this paper, we examine their fundamental properties and explore their connections with maximal ideals and their role in shaping the underlying algebraic structure. Furthermore, we provide several new characterizations that clarify the behavior of these functionals under various algebraic conditions. In addition, we establish and extend a number of related results, offering generalizations and refinements of known theorems in the literature.
Keywords. Banach Algebras, Multiplicative Linear Functionals, Maximal Ideals.

1. Introduction

The concept of multiplicative linear functionals emerged from the development of functional analysis in the early twentieth century, particularly within the theory of Banach algebras. Multiplicative linear functionals are fundamental tools in the study of Banach algebras and related structures. These functionals, often called characters, enabled the identification of the maximal ideal space of a commutative Banach algebra. The theory of multiplicative linear functionals has been extensively developed by various researchers, such as R. Phelps [11]. and Z. Semadeni [12].
W. Zelazko presented some characterizations and properties of multiplicative linear functionals on Banach algebras in a series of articles [14-16]. The topic was further developed through subsequent contributions by mathematicians such as [1, 9]. We recall some standard definitions and results that will be used throughout the paper.

Definition 1.1. Let A be a linear space over K. Let $\|\,.\,\| : A \to K$ be a function such that :

(i) $\| a \| \geq 0$,

(ii) $\| a \| = 0$ if and only if $a = 0$,

(iii) $\| \alpha a \| = | \alpha | \, \| a \|$,

(iv) $\| a+b \| \leq \| a \| + \| b \|$,

for all $a, b \in A, \alpha \in K$.

Then the function $\|\,.\,\|$ is called a *norm* on A and $(A, \|\,.\,\|)$ is called a *normed space.*

Definition 1.2. Let $(A, \|\,.\,\|)$ be a normed space. The sequence $\{a_n\}$ in A is said to *converge* to the point a if for each $\varepsilon > 0$, there exists a positive integer N such that $\| a_n - a \| < \varepsilon$ for all $n > N$.

If the sequence $\{a_n\}$ converges to a, then we write $a_n \to a$ as $n \to \infty$.

The following theorem describes the fundamental properties of convergence in normed spaces

Theorem 1. 3. Let $(A, \|\,.\,\|)$ be a normed space. If $a_n \to a$ $(n \to \infty)$ and $b_n \to b$ $(n \to \infty)$ in A. Then

(i) $a_n \pm b_n \to a \pm b$ $(n \to \infty)$,

(ii) $\lambda a_n \to \lambda a$ $(n \to \infty), \lambda \neq 0$,

(iii) $a_n b_n \to a b$ $(n \to \infty)$.

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Definition 1. 4. Let $(A, \|.\|)$ be a normed space. The sequence $\{a_n\}$ in A is said to *Cauchy* if for each $\varepsilon > 0$, there exists a positive integer N such that $\|a_n - a_m\| < \varepsilon$ for all $n, m > N$.

Definition 1. 5. A *normed algebra* is an algebra A which is a normed linear space $(A, \|.\|)$ such that

$$\|ab\| \leq \|a\| \|b\| \quad (a, b \in A).$$

Definition 1.6 Let A be an algebra equipped with a norm $\|.\|$. Then A is called a *complete normed algebra* if every Cauchy sequence in A converges in A with respect to the norm.

A complete normed algebra A is called *a Banach algebra*.

Definition 1.7. A Banach algebra A is called a *Banach algebra with unit* if A it contains a non-zero element e such that $ea = ae = a \quad (a \in A)$.

A Banach algebra A is called *commutative* if $ab = ba \quad (a, b \in A)$.

Definition 1.8. Let A be a Banach algebra with unit e. An element $a \in A$ is called *invertible* if a has an inverse in A, that is, there exists an element $a^{-1} \in A$ such that $aa^{-1} = a^{-1}a = e$. The element a in A has at most one inverse a^{-1} in A.

Notation. Let A be a Banach algebra with unit e. We shall denote the set of all invertible elements in A by A^{-1}.

2. Multiplicative linear functionals on Banach algebras

In this section, we investigate multiplicative linear functionals on a Banach algebra into in the complex field, and we establish a number of important properties and characterization results associated with them.

Definition 2.1. Let A be a Banach algebra. A linear map $\phi : A \to \mathbb{C}$ is called *a multiplicative linear functional* on A if $\phi \neq 0$ and

$$\phi(ab) = \phi(a)\phi(b) \quad (a, b \in A),$$

that is ϕ is a non-zero homomorphism mapping from A onto $\mathbb{C}$.

Remarks. Let ϕ be a multiplicative linear functional on a Banach algebra A with unit.

(1) The condition $\phi \neq 0$ means that $\phi(a) \neq 0$ for some a of a Banach algebra A.

(2) Let $\lambda_i \in \mathbb{C}$ and $a_i \in A$. By the linearity of ϕ and an induction argument, we have

$$\phi\left(\sum_{i=1}^{n} \lambda_i a_i\right) = \sum_{i=1}^{n} \lambda_i \phi(a_i).$$

(3) Let $a \in A$ and $\lambda \in \mathbb{C}$, we obtain that $\phi(\lambda a) = \lambda \phi(a)$,

In the particular case $\lambda = -1$, the formula simplifies to $\phi(-a) = -\phi(a)$.

We now give some illustrative examples of multiplicative linear functionals.

Examples 2.2

(1) Let X be a compact space. Then $C(X)$ the algebra of all continuous functions from X into $\mathbb{C}$.

Then $C(X)$ is a Banach algebra.

For each $x_0 \in X$, define

$$\phi_{x_0}(f) = f(x_0) \quad (f \in C(X)).$$

Then ϕ_{x_0} is a multiplicative linear functional on $C(X)$.

(2) Let $\mathbb{C}^n = \{(z_1, z_2, \ldots, z_n) : z_i \in \mathbb{C} \ (i = 1, 2, \ldots, n)\}$.

Let $(z_1, z_2, \ldots, z_n), (w_1, w_2, \ldots, w_n) \in \mathbb{C}^n$.

Define multiplication coordinate wise:

$$(z_1, z_2, \ldots, z_n)(w_1, w_2, \ldots, w_n) = (z_1 w_1, z_2 w_2, \ldots, z_n w_n).$$

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Then $\mathbb{C}^n$ is a Banach algebra.
For each index k, define

$$\phi_k(z_1, z_2, \ldots, z_n) = z_k.$$

Then ϕ_k is a multiplicative linear functional on $\mathbb{C}^n$.

(3) Let A be a Banach algebra with unit. Define A^* by

$$A^* = A \oplus \mathbb{C}.$$

Let $a_1, a_2 \in A$ and $\lambda_1, \lambda_2 \in \mathbb{C}$. Let $(a_1, \lambda_1), (a_2, \lambda_2) \in A^*$.

The multiplication in A^* is defined by

$$(a_1, \lambda_1)(a_2, \lambda_2) = (a_1 a_2 + \lambda_1 a_2 + \lambda_2 a_1, \lambda_1 \lambda_2).$$

If A is a Banach algebra with unit, then A^* is a Banach algebra with unit.
Let ϕ be a multiplicative linear functional on A. Define $\phi^* : A^* \to \mathbb{C}$ by

$$\phi^*(a, \lambda) = \phi(a) + \lambda \quad a \in A, (a, \lambda) \in A^*.$$

Let $(a_1, \lambda_1), (a_2, \lambda_2) \in A^*$. Then

$$\begin{aligned}
\phi^*((a_1, \lambda_1)(a_2, \lambda_2)) &= \phi^*(a_1 a_2 + \lambda_1 a_2 + \lambda_2 a_1, \lambda_1 \lambda_2) \\
&= \phi(a_1 a_2 + \lambda_1 a_2 + \lambda_2 a_1) + \lambda_1 \lambda_2 \\
&= \phi(a_1 a_2) + \phi(\lambda_1 a_2) + \phi(\lambda_2 a_1) + \lambda_1 \lambda_2 \\
&= \phi(a_1)\phi(a_2) + \lambda_1 \phi(a_2) + \lambda_2 \phi(a_1) + \lambda_1 \lambda_2 \\
&= (\phi(a_1) + \lambda_1)(\phi(a_2) + \lambda_2) \\
&= \phi^*(a_1, \lambda_1)\ \phi^*(a_2, \lambda_2).
\end{aligned}$$

Then ϕ^* is a multiplicative linear functional on A^*.
The next theorem guarantees the existence of non-zero multiplicative linear functionals on a Banach algebra."
Theorem 2.3. [4]. *Let A be a commutative Banach algebra with unit. Then there exists at least one multiplicative linear functional ϕ on A.*
Notation. Let A be a commutative Banach algebra with unit. Then Φ_A denotes the set of all multiplicative linear functionals on A.
The following example aims to illustrate this notation:
we define $\phi : \mathbb{C} \to \mathbb{C}$ by

$$\phi(z) = z \quad (z \in \mathbb{C}).$$

Then ϕ is a multiplicative linear functional mapping on $\mathbb{C}$.
Thus $\Phi_{\mathbb{C}} = \{z\}$. That is, $\Phi_{\mathbb{C}}$ contains only one point.

The next theorems give the basic properties of multiplicative linear functionals on Banach algebras. The proofs follow by standard techniques, see [7].
Theorem 2.4. *Let A be a* commutative *Banach algebra A with unit e and* let $\phi \in \Phi_A$. *Then*

(i) $\phi(0) = 0$
(ii) $\phi(e) = 1$
(iii) $\phi(\lambda) = \lambda \quad (\lambda \in \mathbb{C})$
(iv) $\phi(a^n) = (\phi(a))^n \quad (n \in \mathbb{N}, a \in A)$.

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(v) if $\phi(a) = a \ (a \in A)$, *then* $\phi(a^n) = a^n \ (n \in \mathbb{N}, a \in A)$.

Remark. In particular, if $e = 1$ in (ii) of Theorem 2.4, then $\phi(1) = 1$.

Theorem 2.5. *Let* A *be a* commutative *Banach algebra* A *with unit and* let $\phi \in \Phi_A$. *Let* $a, b \in A$. *Then*

(i) $\phi(\phi(a)) = \phi(a)$,
(ii) $\phi(a - \phi(a)) = 0$,
(iii) $\phi((a - \phi(a)) \pm (b - \phi(b))) = 0$,
(iv) $\phi((a - \phi(a))\ (b - \phi(b))) = 0$.

Theorem 2.6. *Let* A *be a* commutative *Banach algebra* A *with unit and* let $\phi \in \Phi_A$. *Let* $a, b \in A$. *Then*

(i) *if* $a \in A^{-1}$, *then* $\phi(a) \neq 0$,
(ii) $\phi(a^{-1}) = \phi((a))^{-1}$,
(iii) $\phi(\frac{a}{b}) = \phi((a))\ \phi((b))^{-1} \quad (b \neq 0)$,
(iv) $\phi\ (ab)^{-1} = \phi((b))^{-1}\phi((a))^{-1}$.

Theorem 2.7. [5] *Let* A *be a* commutative *Banach algebra* A *with unit and* let $\phi \in \Phi_A$. *Then* ϕ *is continuous and* $\| \phi \| = 1$.

The following theorem concerns the convergence of multiplicative linear functionals on Banach algebras.

Theorem 2.8. *Let* A *be a* commutative *Banach algebra* A *with unit and* let $\phi \in \Phi_A$. *Let* $a, b \in A$. *Then*

(i) if $a_n \to a \ (n \to \infty)$ in A, then $\phi(a_n) \to \phi(a)$,
(ii) if $a_n \to a \ (n \to \infty)$ and $b_n \to b \ (n \to \infty)$ in A, then $\phi(a_n \pm b_n) \to \phi(a \pm b)$,
(iii) if $a_n \to a \ (n \to \infty)$ in A, then $\phi(\lambda a_n) \to \phi(\lambda a) \quad (\lambda \in \mathbb{C})$,
(iv) if $a_n \to a \ (n \to \infty)$ and $b_n \to b \ (n \to \infty)$ in A, then $\phi(a_n b_n) \to \phi(ab)$.

Proof .We prove (i) and (ii) . The proofs of the remaining parts are analogous.

(i) Let $a_n \to a \ (n \to \infty)$. Then for each $\varepsilon > 0$, there exists a positive integer N such that

$\| a_n - a \| < \varepsilon$ for all $n > N$.

We have

$$\| \phi(a_n) - \phi(a) \| = \| \phi(a_n - a) \| \le \| \phi \| \, \| a_n - a \| < \varepsilon \, \| \phi \| .$$

Since $\| \phi \| = 1$, it follows that $\| \phi(a_n) - \phi(a) \| < \varepsilon$ for all $n > N$.

Thus $\phi(a_n) \to \phi(a) \ (n \to \infty)$.

(ii) Let $a_n \to a \ (n \to \infty)$ and $b_n \to b \ (n \to \infty)$ in A. Then

$a_n \pm b_n \to a \pm b$ (Theorem (ii)).

It follows from (i) that $\phi(a_n \pm b_n) \to \phi(a \pm b)$.

Theorem 2.9. *Let* A *be a commutative Banach algebra* A *with unit and* let $\phi \in \Phi_A$. *Let* $a, b \in A$. *Then there exists an element* b *in* A *with* $\phi(b) = 0$.

Proof. Let $\phi \in \Phi_A$ with $\phi(a) \neq 0$ for some $a \in A$. Then $\frac{a}{\phi(a)} \in A$.

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Let $b = 1 - \dfrac{a}{\phi(a)}$. Then $b \in A$.

Since $\phi(a) \neq 0$, so there exists an invertible element $(\phi(a))^{-1}$ in A .

Therefore

$$\phi\left(\frac{a}{\phi(a)}\right) = \phi(a\,(\phi(a))^{-1})$$
$$= \phi(a\,(\phi(a^{-1})))$$
$$= \phi(a)\,\phi(\phi(a^{-1}))$$
$$= \phi(a)\,\phi(a^{-1})$$
$$= \phi(a\,a^{-1})$$
$$= \phi(e).$$

So, we have

$$\phi(b) = \phi(1) - \phi\left(\frac{a}{\phi(a)}\right)$$
$$= \phi(1) - \phi(e)$$
$$= 1 - 1$$
$$= 0.$$

This completes the proof.

Remark: The proof of Theorem 2.9 yields, $\phi\left(\dfrac{a}{\phi(a)}\right) = \phi(e)$, if $e = 1$, then $\phi\left(\dfrac{a}{\phi(a)}\right) = \phi(1) = 1$.

So we have $\phi\left(\dfrac{a}{\phi(a)}\right) = 1$.

Theorem 2.10. *Let* A *be a* commutative *Banach algebra* A *with unit. Let* $\phi_1, \phi_2 \in \Phi_A$ *and* $c \neq 0$. If $\phi_1 = c\,\phi_2$, *then* $c = 1$.

Proof. We present two methods of proof.

Method (I) : Let $a \in A$. Then

$$\phi_1(a^2) = (\phi_1(a))^2 \text{ (Theorem 2.4 (iv), } n = 2\text{)}$$
$$= c^2(\phi_2(a))^2$$
$$= c^2\phi_2(a^2).$$

Hence $c\,\phi_2(a^2) = c^2\phi_2(a^2)$.

Thus $c = c^2$. Since $c \neq 0$, it follows that $c = 1$.

Method (II) : Let $\phi_1 = c\,\phi_2$. Then $\phi_1(1) = c\,\phi_2(1)$.

Since $\phi_1(1) = \phi_2(1) = 1$ (Theorem 2.2 (ii)), so $c = 1$.

Lemma 2.11. *Let* A *be a commutative Banach algebra with unit. Let* $\phi_1, \phi_2 \in \Phi_A$. *Then* ϕ_1 *and* ϕ_2 *are linear independent.*

Proof. Let $\phi_1, \phi_2 \in \Phi_A$ and let $a, b \in A$ with $a \neq b$.

Assume $a\,\phi_1 + b\,\phi_2 = 0$. Then $\phi_1 = -\dfrac{b}{a}\,\phi_2$.

It follows that $\phi_1(1) = -\dfrac{b}{a}\,\phi_2(1)$. Thus $a = -b$ which implies $b\,(\phi_2 - \phi_1) = 0$.

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Since $\phi_2 - \phi_1 \neq 0$, so $b = 0$ and follows that $a = 0$.

Hence ϕ_1 and ϕ_2 are linear independent.

Theorem 2.12. *Let A be a commutative Banach algebra with unit. Let $\phi_1, \phi_2, \dots, \phi_n \in \Phi_A$. Then $\phi_1, \phi_2, \dots, \phi_n$ are linear independent.*

Proof. The proof follows by induction on n.

Proposition 2.13. *Let A be a commutative Banach algebra with unit and let $\phi \in \Phi_A$. Then there are no $a, b \in A$ such that $\phi(a) = 1$ and $a + ab = b$.*

Proof. On contrary, suppose that there exists $a, b \in A$ such that $\phi(a) = 1$ *and* $a + ab = b$.

Therefore
$$\begin{aligned} 1 + \phi(b) &= \phi(a) + \phi(a)\phi(b) \\ &= \phi(a + ab) \\ &= \phi(b). \end{aligned}$$

Thus $1 + \phi(b) = \phi(b)$ which is impossible.

This contradiction shows that the assumption is false.

Hence the proof is complete.

In the study of Banach algebras, every non-zero multiplicative linear functional on a commutative Banach algebra with unit determines a maximal ideal, defined as follows :

Definition 2.14. Let A be a commutative Banach algebra and let $\phi \in \Phi_A$. The *maximal ideal* with ϕ in A, denoted by M_ϕ, is defined by

$$M_\phi = \{ a \in A : \phi(a) = 0 \}.$$

We give the following observations about the maximal ideal of multiplicative linear functionals ϕ on A.

(i) Clearly $0 \in M_\phi$ because $\phi(0) = 0$ and so M_ϕ is a non-empty set.

(ii) Since $\phi \neq 0$ and $\phi(1) = 1$, so we have $1 \notin M_\phi$. Thus $M_\phi \neq A$.

(iii) The continuity of ϕ implies that M_ϕ is a closed set.

Example 2.15. Let $A = \mathbb{C}^n$ and let $(z_1, z_2, \dots, z_n) \in \mathbb{C}^n$.

As in example 1 we define the multiplicative linear functional on $\mathbb{C}^n$ by :

$$\phi_k(z_1, z_2, \dots, z_n) = z_k.$$

Then we have

$$\begin{aligned} M_\phi &= \{ (z_1, z_2, \dots, z_n) \in \mathbb{C}^n : \phi_k(z_1, z_2, \dots, z_n) \} \\ &= \{ (z_1, z_2, \dots, z_n) \in \mathbb{C}^n : z_k = 0 \}. \end{aligned}$$

We present and prove several results related to maximal ideal spaces.

Proposition 2.16. *Let A be a commutative Banach algebra with unit and* let $\phi \in \Phi_A$. Then

(i) $ax \in M_\phi$ $(a \in A,\ x \in M_\phi)$

(ii) $a - \phi(a) \in M_\phi$ $(a \in A)$.

Proof. (i) Let $a \in A$ and $x \in M_\phi$. Then

$$\phi(ax) = \phi(a)\,\phi(x) = 0.$$

Thus $ax \in M_\phi$.

(ii) Let $a \in A$. Then

$$\phi(a - \phi(a)) = \phi(a) - \phi(\phi(a))$$

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$$= \phi(a) - \phi(a)$$
$$= 0.$$

Thus $a - \phi(a) \in M_{\phi}$.

Lemma 2.17. *Let A be a commutative Banach algebra with unit and $\phi \in \Phi_A$. Let $a \in A$. Assume that $x \in A \setminus M_{\phi}$. Then* $\phi\left(\frac{\phi(a)}{\phi(x)} x\right) = \phi(a)$.

Proof. Let $\phi(a) = \lambda$ ($\lambda \in \mathbb{C}$). Then

$$\begin{aligned}
\phi\left(\frac{\phi(a)}{\phi(x)} x\right) &= \phi\left(\frac{\lambda}{\phi(x)} x\right) \\
&= \lambda\, \phi\left(\frac{1}{\phi(x)} x\right) \\
&= \lambda . 1 \\
&= \lambda \\
&= \phi(a).
\end{aligned}$$

Hence $\phi\left(\frac{\phi(a)}{\phi(x)} x\right) = \phi(a)$.

Lemma 2.18. *Let A be a commutative Banach algebra with unit e and let $\phi \in \Phi_A$. Then*

$$a - \frac{\phi(a)}{\phi(x)} x \in M_{\phi} \quad (a \in A,\ x \in A \setminus M_{\phi}).$$

Proof. Let $a \in A$. Then

$$\begin{aligned}
\phi\left(a - \frac{\phi(a)}{\phi(x)} x\right) &= \phi(a) - \phi\left(\frac{\phi(a)}{\phi(x)} x\right) \\
&= \phi(a) - \phi(a) \\
&= 0.
\end{aligned}$$

Thus $a - \frac{\phi(a)}{\phi(x)} x \in M_{\phi}$.

Theorem 2.19. *Let A be a commutative Banach algebra with unit and let $\phi \in \Phi_A$. Let $a, j \in A$.*

(i) *Then $\phi(j) = 1$ if and only if $a - aj \in M_{\phi}$.*

(ii) *If $\phi(j) = 1$, then $\phi(a + aj) = 2\phi(a)$.*

Proof. Let $a, j \in A$. Then $a - aj \in A$.

We have

$$\begin{aligned}
\phi(a - aj) &= \phi(a) - \phi(aj). \\
&= \phi(a) - \phi(a)\phi(j) \\
&= \phi(a) - \phi(a) \\
&= 0.
\end{aligned}$$

Hence $a - aj \in M_{\phi}$.

Conversely, let $a, j \in A$ such that $a - aj \in M_{\phi}$.

Then $\phi(a - aj) = 0$ and so $\phi(a) - \phi(a)\,\phi(j) = 0$.

Thus $\phi(a)(1 - \phi(j)) = 0$.

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Since $\phi(a) \neq 0$ for some $a \in A$, so $1-\phi(j)=0$.
Therefore $\phi(j)=1$.
(ii) Let $a, j \in A$. Then $a + a j \in A$.

$$\begin{aligned}\phi(a+a j) &= \phi(a) + \phi(a j)\\ &= \phi(a) + \phi(a)\,\phi(j)\\ &= \phi(a) + \phi(a)\\ &= 2\,\phi(a).\end{aligned}$$

Remark Let A be a commutative Banach algebra with unit and let $\phi \in \Phi_A$. Let $a, j \in A$.
Note that $a + a j \notin M_{\phi}$.

Theorem 2.20. *Let* A *be a commutative Banach algebra with unit and let* $\phi \in \Phi_A$. *Then*

$$A = M_{\phi} + \mathbb{C}\cdot 1.$$

Proof. Firstly, we show that $A \subset M_{\phi} + \mathbb{C}\cdot 1$.

Let $a \in A$. Then there uniquely $\phi(a) \in \mathbb{C}\cdot 1$ such that $a - \phi(a) \in M_{\phi}$ (Theorem 2. 16 (ii)).

We can write $a = a - \phi(a) + \phi(a)$.

Since $(a - \phi(a)) + \phi(a) \in M_{\phi} + \mathbb{C}\cdot 1$, so $a \in M_{\phi} + \mathbb{C}\cdot 1$.

Secondly, we show that $M_{\phi} + \mathbb{C}\cdot 1 \subset A$.

Since $M_{\phi} \subset A$ and $\mathbb{C}\cdot 1 \subset A$. So $M_{\phi} + \mathbb{C}\cdot 1 \subset A$.

Thus $A = M_{\phi} + \mathbb{C}\cdot 1$.

This completes the proof.

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