

Original article

# Spectral Analysis of Signals Using Fourier Transforms: Frequency Spectrum, Power Spectrum, and Energy Spectrum

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## Abstract

Signal analysis in the frequency domain occupies a central position in applied mathematics and engineering. This work presents a structured examination of three fundamental spectrum types derived through Fourier transform theory: the frequency spectrum, the power spectrum, and the energy spectrum. For each type, precise mathematical definitions are established, computational procedures are outlined, and illustrative numerical examples are worked through using the Continuous Fourier Transform (CFT), the Discrete Fourier Transform (DFT), and the Fast Fourier Transform (FFT). The internal relationships among these three spectral representations are discussed, and Parseval's theorem is invoked to validate the equivalence between energy quantified in the time domain and energy quantified in the frequency domain. All numerical results obtained through digital computation accord with analytically derived expectations, confirming the theoretical framework and illustrating the broad engineering utility of spectral methods.

**Keywords.** Fourier Transform, Frequency Spectrum, Power Spectrum, Energy Spectrum, DFT, FFT.

## Introduction

Spectral methods trace their roots to Isaac Newton's 1666 prism experiment, in which he demonstrated that white sunlight separates into a band of colors on refraction, what is now called the visible spectrum. The Latin root "spectrum", meaning "appearance" or "image", entered scientific vocabulary at that time. During the nineteenth century, Joseph von Fraunhofer's mapping of dark absorption lines across the solar spectrum, and the subsequent identification by Kirchhoff and Bunsen of element-specific spectral signatures in the 1860s, elevated spectroscopy into a rigorous quantitative discipline.

The theoretical apparatus linking spectra to mathematical analysis was constructed by Joseph Fourier, who in 1807 showed that periodic functions could be represented as superpositions of trigonometric terms. Extensions to non-periodic signals followed through the Fourier transform, and the Sturm-Liouville generalization of 1836 broadened the orthogonal-expansion framework. A pivotal algorithmic advance came in 1965 when Cooley and Tukey demonstrated that the discrete Fourier transform could be computed at a cost of  $O(N \log N)$  rather than  $O(N^2)$ , transforming Fourier methods from a theoretical tool into a practical workhorse for large-scale digital signal processing.

When a signal undergoes a Fourier transform, the resulting frequency-domain representation is its spectrum. Depending on the nature of the signal and the quantity of interest, three distinct spectral representations are relevant. The frequency spectrum reveals which frequencies are present and with what amplitude. The power spectrum describes how signal power is partitioned across frequency bands and is most natural for periodic or stationary random processes. The energy spectrum characterizes the frequency-domain distribution of total energy and is appropriate for transient, finite-energy signals. This paper develops all three representations systematically, supporting the derivations with worked numerical examples and verifying results against known analytical expressions.

## Theoretical Background: Fourier Transforms

### The Fourier Transform and Its Inverse

Given a function  $f(t)$  that satisfies the Dirichlet conditions chiefly absolute integrability over the real line its Fourier transform  $F(\omega)$  is:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

The original function is recovered via the inverse transform:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

The Fourier series addresses strictly periodic signals by expanding them in a discrete set of harmonically related sinusoids, producing a line spectrum. The Fourier transform generalizes this analysis to aperiodic signals defined over the

entire real line and yields a continuous spectrum. This contrast between periodic (discrete) and aperiodic (continuous) spectral representations underpins the categorization of spectrum types discussed in subsequent sections.

### Principal Properties

Several properties of the Fourier transform are routinely exploited in spectral analysis:

- Linearity:  $\mathcal{M}\{af(t) + bg(t)\} = aF(\omega) + bG(\omega)$ , where  $a$  and  $b$  are constants.
- Time shifting:  $\mathcal{M}\{f(t - t_0)\} = e^{-i\omega t_0} F(\omega)$ .
- Frequency shifting:  $\mathcal{M}[e^{i\omega t_0} f(t)] = F(\omega - \omega_0)$ .
- Scaling:  $\mathcal{M}[f(at)] = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$ .
- Differentiation:  $\mathcal{M}\{d^n f/dt^n\} = (j\omega)^n F(\omega)$ .
- Convolution theorem:  $\mathcal{M}(f * g)(t) = F(\omega) \cdot G(\omega)$ .

### Frequency Spectrum

#### Definition

The frequency spectrum  $|X(f)|$  of a signal  $x(t)$  is the magnitude of its Fourier transform, quantifying the amplitude contributed by each frequency component:

$$|X(f)| = \left| \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \right|$$

For digital signals of finite length  $N$ , the Discrete Fourier Transform supplies an equivalent finite-resolution representation:

$$X[m] = \sum_{n=0}^{N-1} x[n] w_N^n \quad n = 0, 1, \dots, N-1$$

Whereas time-domain analysis traces how a signal changes instant by instant, frequency-domain analysis decomposes it into its constituent oscillatory components, making it easier to identify dominant frequencies, characterize bandwidth, and detect interfering tones embedded in noise.

#### Numerical Approximation via the Trapezoidal Rule

When a closed-form antiderivative of  $x(t) \cdot e^{-i\omega t}$  is unavailable, the integral is approximated numerically. The trapezoidal rule over an interval  $[a, b]$  subdivided into  $N$  equal steps of width  $\Delta t = \frac{(b-a)}{N-1}$  is:

$$t_n = a + n \Delta t, \text{ for } n = 0, 1, \dots, N.$$

$$F(\omega) \approx \Delta t \left[ \frac{f(t_0)e^{-i\omega t_0}}{2} + \sum_{n=1}^{N-1} f(t_n)e^{-i\omega t_n} + \frac{f(t_N)e^{-i\omega t_N}}{2} \right]$$

Accuracy improves as  $N$  increases and deteriorates when the integrand contains sharp discontinuities or rapid oscillations.

#### Worked Examples

Example 3.1: Rectangular pulse. For  $p(t) = 1$  when  $|t| \leq a$  and  $p(t) = 0$  elsewhere, direct integration yields:

$$P(\omega) = \frac{2\sin\omega a}{\omega} = 2a \operatorname{sinc}(\omega a)$$

Setting  $a = 2$  gives the analytical result  $P(\omega) = \frac{2\sin(2\omega)}{\omega}$ . Evaluating at

$\omega = \left[0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right]$  using four DFT samples  $x[n] = [1, 1, 1, 1]$  reproduces the same values numerically:

| Frequency $\omega$ (rad/s) | $ P(\omega) $ — analytical | $ X[m] $ — DFT |
|----------------------------|----------------------------|----------------|
| 0                          | 4                          | 4              |
| $\pi/2$                    | 0                          | 0              |
| $\pi$                      | 0                          | 0              |
| $3\pi/2$                   | 0                          | 0              |

The spectral energy is confined entirely to the zero-frequency (DC) component, consistent with the uniform unit amplitude of the pulse.

Example 3.2: Double-sided exponential. For  $f(t) = \begin{cases} e^{at} & \text{if } t < 0 \\ e^{-at} & \text{if } t > 0 \end{cases}$

with  $a > 0$ , the transform is:

$$F(\omega) = \frac{2a}{a^2 + \omega^2}$$

With  $a = 1$  and the trapezoidal rule applied over  $[-1.5, 1.5]$  at  $N = 5$  points, the approximated magnitudes at the chosen frequencies are

$$|F(0)| = 1.6259, |F(\pi/2)| = 0.9028, |F(\pi)| = 0.2490, \text{ and } |F(3\pi/2)| = 0.2137.$$

**Example 3.3** Sine function. Expressed as

$\sin(x) = [e^{ix} - e^{-ix}] / 2i$ , the continuous transform evaluates to:

$$F(\omega) = -i\pi [\delta(\omega + 1) + \delta(\omega - 1)]$$

Because  $\sin(t)$  carries infinite energy over  $(-\infty, \infty)$ , direct numerical integration is impractical. Sampling one period with  $X[n] = \sin\left(\frac{2\pi n}{4}\right)$  for  $n = 0, 1, 2, 3$  yields  $x[n] = [0, 1, 0, -1]$ , and the DFT gives  $X[m] = [0, -2i, 0, +2i]$ , or in magnitude  $|X[m]| = [0, 2, 0, 2]$ . The two nonzero entries at indices 1 and 3 correspond to the positive and negative frequency components of the sine wave.

## Applications

- Communications: spectral planning for carrier allocation and interference mitigation in wireless systems.
- Biomedical engineering: frequency-band analysis of EEG and ECG recordings to identify pathological signatures.
- Acoustics: decomposition of complex audio mixtures and noise characterization in recordings.
- Radar and remote sensing: Doppler-shift extraction to determine target velocity and bearing.
- Image processing: spatial-frequency decomposition for compression and edge enhancement.
- Astronomy: stellar velocity estimation from Doppler shifts in electromagnetic emission spectra.

## Power Spectrum

### Definition

The power spectrum  $P(f)$  measures the distribution of signal power across frequency. It is the preferred spectral representation for periodic signals and stationary stochastic processes, which generally lack finite total energy but possess a well-defined time-averaged power. Two related formulations arise in practice:

Non-normalized (total) power spectrum:

$$P[m] = |X[m]|^2$$

Power Spectral Density (PSD), normalized per sample:

$$PSD[m] = (1/N) |X[m]|^2$$

The PSD is scale-invariant with respect to record length and therefore facilitates meaningful comparisons between spectral estimates drawn from signals of differing durations.

## Worked Examples

Example 4.1 Rectangular pulse. Starting from  $X[m] = [4, 0, 0, 0]$ :

| $\omega$ | $ X[m] $ | $P[m] =  X[m] ^2$ | $PSD = P[m]/4$ |
|----------|----------|-------------------|----------------|
| 0        | 4        | 16                | 4              |
| $\pi/2$  | 0        | 0                 | 0              |
| $\pi$    | 0        | 0                 | 0              |
| $3\pi/2$ | 0        | 0                 | 0              |

Power is exclusively at the DC term, reflecting the constant-valued nature of the pulse over its support.

Example 4.2 Double-sided exponential. With DFT magnitudes

$$|X[m]| = [1.553, 0.9214, 0.7176, 0.9214]:$$

| $\omega$ | $ X[m] $ | $P[m]$ | $PSD = P[m]/4$ |
|----------|----------|--------|----------------|
| 0        | 1.553    | 2.411  | 0.6030         |
| $\pi/2$  | 0.9214   | 0.8489 | 0.2122         |
| $\pi$    | 0.7176   | 0.5149 | 0.1287         |
| $3\pi/2$ | 0.9214   | 0.8489 | 0.2122         |

The monotonically decreasing spectral magnitude from DC reflects the exponential amplitude decay in the time domain.

Example 4.3 Sine function. With  $|X[m]| = [0, 2, 0, 2]$ :

| $\omega$ | $ X[m] $ | $P[m]$ | $PSD = P[m]/4$ |
|----------|----------|--------|----------------|
| 0        | 0        | 0      | 0              |
| $\pi/2$  | 2        | 4      | 1              |
| $\pi$    | 0        | 0      | 0              |
| $3\pi/2$ | 2        | 4      | 1              |

The symmetric power distribution at indices 1 and 3 encodes the two complex-exponential components of the sine function.

### Applications

- Frequency-component analysis and dominant-frequency identification.
- Noise floor estimation and interference localization in sensor and communication systems.
- Channel characterization and filter design in digital communications.
- Vibration monitoring and predictive maintenance via mechanical resonance detection.
- Biomedical signal analysis for periodic physiological rhythms (cardiac, neural, respiratory).

## Energy Spectrum

### Definition and Parseval's Theorem

The energy spectrum  $E(\omega)$  quantifies how the total energy of a finite-energy signal is allocated across frequencies. A signal  $x(t)$  is classified as an energy signal if:

$$\int_{-\infty}^{\infty} |X(t)|^2 dt < \infty$$

Under this condition, the energy spectrum is:

$$E(\omega) = |X(\omega)|^2$$

Parseval's theorem states that total energy is invariant between the two domains:

$$E_{\text{total}} = \int_{-\infty}^{\infty} |X(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = E_{\text{freq}}$$

For a finite-length discrete sequence of  $N$  samples, the discrete form of Parseval's relation is:

$$E_{\text{freq}} = \frac{1}{N} \sum_{m=0}^{N-1} |X[m]|^2 = E_{\text{time}} = \sum_{m=0}^{N-1} |x[m]|^2$$

This identity serves as an algebraic consistency check whenever both domain energies are computed from the same dataset.

### Computation Steps

1. Compute the DFT or FFT of the finite-length signal to obtain  $X[m]$ .
2. Form the energy spectrum  $P[m] = |X[m]|^2$  at each frequency index  $m$ .
3. Sum and normalize to obtain the frequency-domain energy:

$$E_{freq} = (1/N) \sum P[m].$$

4. Independently compute the time-domain energy:  $E_{time} = \sum |x[n]|^2$ .
5. Verify  $E_{time} = E_{freq}$  to confirm computational correctness.

### Worked Examples

Example 5.1 Rectangular pulse. For  $x[n] = [1, 1, 1, 1]$  ( $N = 4$ ):

DFT output  $X[m] = [4, 0, 0, 0]$  yields energy spectrum  $P[m] = [16, 0, 0, 0]$ , so:

$$E_{freq} = (1/4)(16 + 0 + 0 + 0) = 4 \rightarrow E_{time} = E_{freq} = 4$$

Example 5.2 Double-sided exponential. For  $x[n] = [1, 0.3679, 0.1353, 0.0498]$ :

From  $X[m] = [1.553, 0.9214, 0.7176, 0.9214]$  and

$P[m] = [2.4118, 0.8489, 0.5149, 0.8489]$ :

$$E_{freq} = (1/4)(2.4118 + 0.8489 + 0.5149 + 0.8489) = 1.156$$

Example 5.3 Sine function. For  $x[n] = [0, 1, 0, -1]$  ( $N = 4$ ):

DFT output  $X[m] = [0, -2i, 0, +2i]$  yields  $P[m] = [0, 4, 0, 4]$ , giving:

$$E_{freq} = (1/4)(0 + 4 + 0 + 4) = 2 \rightarrow E_{time} = E_{freq} = 2$$

Important caveat: the continuous sine function over an infinite interval is an infinite-energy signal; energy spectral analysis is meaningful only for the finite-length discrete sample used here.

### Applications

- Non-periodic signal analysis: identifying frequency bands that carry the bulk of signal energy.
- Time-invisible component detection: discovering transient features that are not apparent in the time waveform.
- Filter synthesis: designing pass-bands aligned with energy-bearing frequency regions.
- Data compression: discarding frequency bins whose energy falls below a perceptibility threshold.
- Biomedical engineering: localizing anomalous energy concentrations in ECG and EEG recordings.
- Structural health monitoring: tracking energy spectral shifts indicative of progressive mechanical fatigue.

### Comparative Summary of the Three Spectrum Types

| Property                | Frequency Spectrum       | Power Spectrum              | Energy Spectrum             |
|-------------------------|--------------------------|-----------------------------|-----------------------------|
| Formula                 | $ X(f) $                 | $ X(f) ^2$                  | $ X(f) ^2$ (finite energy)  |
| Applicable signal class | Any signal               | Periodic/stationary random  | Non-periodic, finite-energy |
| Normalization           | None                     | Optional: $1/N$ for PSD     | $1/N$ for $E_{freq}$        |
| Governing theorem       | Fourier inversion        | Wiener–Khinchin             | Parseval’s theorem          |
| Primary purpose         | Frequency identification | Power distribution analysis | Energy budget verification  |

All three spectral forms are rooted in the squared magnitude of the Fourier transform but address different physical questions. The choice among them is governed by the signal class and the analytical objective: identifying which frequencies exist, understanding how power is shared among them, or accounting for total signal energy.

## Conclusion

A unified treatment of three foundational spectral representations the frequency spectrum, the power spectrum, and the energy spectrum has been developed through Fourier transform theory and illustrated with numerical examples drawn from three canonical signal types: the rectangular pulse, the double-sided exponential, and the sine function.

Although all three spectral types are derived from the magnitude squared of the Fourier transform, they are conceptually distinct and suited to different signal classes. The frequency spectrum is the broadest descriptor, applicable to any transformable signal. The power spectrum is the natural tool for periodic and stationary random signals whose total energy is unbounded. The energy spectrum is reserved for finite-energy aperiodic signals, where Parseval's theorem provides an exact algebraic link between time-domain and frequency-domain energy totals, a link confirmed numerically in all three examples.

From a computational standpoint, the DFT and FFT deliver identical spectral values, but the FFT achieves this at a computational cost of  $O(N \log N)$  compared with  $O(N^2)$  for the direct DFT formula. For signals whose continuous Fourier transform cannot be evaluated in closed form or integrated numerically over an infinite domain, exemplified here by the sine function, the DFT on a finite-duration sample provides a practical and theoretically grounded alternative.

The breadth of applications documented for each spectrum type, spanning wireless communications, biomedical signal processing, acoustic engineering, radar, image analysis, and astrophysics, confirms that spectral analysis remains among the most consequential tools available to working scientists and engineers. Extensions to two-dimensional signals, multi-resolution wavelet representations, and nonstationary time-frequency methods represent natural directions for future work.

**Conflict of interest.** Nil

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