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Original article

β-compactness in Pythagorean fuzzy topological spaces

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Abstract
In this paper, we introduce a new type of compact spaces called β-compact spaces in a Pythagorean fuzzy topological space, using Pythagorean fuzzy β-open sets, we study some of their properties and discuss the relationship between these spaces and other types of compactness. We have reached many results, such as every Pythagorean fuzzy strongly compact space X which satisfy the condition "every Pythagorean fuzzy β-open set of X is denes" is compact.
Keywords: Pythagorean fuzzy, Pythagorean fuzzy β-open set, Pythagorean fuzzy β-compact space.

Introduction

The concept of fuzzy sets was introduced by Zadeh in his classic paper [1]. The theory of fuzzy topological spaces was introduced and developed by Chang [2]. Atanassov [3] introduced the notion of intuitionistic fuzzy sets, and Coker [4] introduced the intuitionistic fuzzy topological spaces. Yader [5,6] introduced the notion of Pythagorean fuzzy subsets, which is a new class of non-standard fuzzy subsets and which has many effective applicants natural and social sciences [7]. Compactness in Pythagorean fuzzy topological spaces was introduced by Haydar. This paper aims to study the concept of Pythagorean fuzzy β-compact space, Pythagorean fuzzy Strongly compact space, Pythagorean fuzzy quasi H-closed space and Pythagorean fuzzy Extremally disconnected topological space. In [8] we study the relation between sets in a Pythagorean fuzzy topological space. We know that every Pythagorean fuzzy preopen set is Pythagorean fuzzy β-open set, in this paper, we studied the necessary and sufficient condition that makes every Pythagorean fuzzy β-open set is a Pythagorean fuzzy preopen set.

1- Preliminaries:

We recall some basic notions of Intuitionistic fuzzy topology spaces and Pythagorean fuzzy Topological Spaces.

Definition 2-1:[3]

Let X be a non-empty fixed set. An intuitionistic fuzzy set (IFS for short) A in X is defined by $A = \{\langle x, \mu_A(x), v_A(x)\rangle : x \in X\}$. Where the functions $\mu_A: X \to [0, 1]$ and $v_A: X \to [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $v_A(x)$) of each element $x \in X$ to the set A, respectively. And $0 \le \mu_A(x) + v_A(x) \le 1$ for each $x \in X$. Denote by IFS (X) the set of all intuitionistic fuzzy sets in X.

Definition 2-2: [5,6]

A Pythagorean fuzzy subset A of a non-empty set X is a pair $A = (\mu_A, \vartheta_A)$ of a membership function $\mu_A: X \to [0,1]$ and a non-membership function $\vartheta_A: X \to [0,1]$, $0 \le \mu_A^2(x) + \vartheta_A^2(x) \le 1$ for each element $x \in X$. Supposing $\mu_A^2(x) + \vartheta_A^2(x) \le 1$, then there is a degree of indeterminacy of $x \in X$ to A defined by $\pi_A(x) = \sqrt{1 - [\mu_A^2(x) + \vartheta_A^2(x)]}$ and $\pi_A(x) \in [0,1]$. We denote the set of all Pythagorean fuzzy subsets over X by $PFS(X)$.

Definition 2-3: [5,6]

Let $A = (\mu_A, \vartheta_A)$ and $B = (\mu_B, \vartheta_B)$ be two Pythagorean fuzzy subsets of a set X. Then,

i- The complement of A is defined by $A^c = (\vartheta_A, \mu_A)$.
ii- $A \cup B = (max\{\mu_A, \mu_B\}, min\{\vartheta_A, \vartheta_B\})$.
iii- $A \cap B = (min\{\mu_A, \mu_B\}, max\{\vartheta_A, \vartheta_B\})$.
iv- We say A is a subset of B if $\mu_A \ge \mu_B$ and $\vartheta_A \le \vartheta_B$, for each $x \in X$ and write $A \subseteq B$..

Definition 2-4: [7]

Let X be a non-empty fixed set, and let τ be a family of Pythagorean fuzzy subsets of X. If

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i- $0,1 \in \tau$.
ii- For any $A_1, A_2 \in \tau$, we have $A_1 \cap A_2 \in \tau$.
iii- $\bigcup_{i \in I} A_i \in \tau$ for any arbitrary family $\{A_i; i \in I\} \subseteq \tau$.

In this case, the pair (X, τ) is called a Pythagorean fuzzy topological space (PFTS for short), and each Pythagorean fuzzy subset in τ is known as a Pythagorean fuzzy open set (PFOS for short) in X. The complement of a Pythagorean fuzzy open subset is called a Pythagorean fuzzy closed subset.

Definition 2-5: [9]

Let (X, τ) is a Pythagorean fuzzy topological space and A be a Pythagorean fuzzy subset in X. Then the fuzzy interior and fuzzy closure of A are defined by

$$int\ (A) = \cup\{U; U\ is\ a \text{ Pythagorean fuzzy open subset in } X\ and\ U \subseteq A\}$$

And

$$cl\ (A) = \cap\{K; K\ is\ a \text{ Pythagorean fuzzy closed subset in } X\ and\ A \subseteq K\}$$

Definition 2-6: [8]

Let A be a PFS in a PFTS X. A is called a Pythagorean fuzzy β-open set (PFβOS in short) of X if $A \subseteq cl(int(cl(A)))$.The complement of a Pythagorean fuzzy β-open (semi preopen) set (PFβOS in short) is called a Pythagorean fuzzy β-closed set (PFβCS in short) of X.

Definition 2-7: [10]

Let A be a PFS in a PFTS X. A is called a Pythagorean fuzzy α-open set (PFαOS in short) of X if $A \subseteq int(cl(int(A)))$. The complement of a Pythagorean fuzzy α-open set is called a Pythagorean fuzzy α-closed set (PFαCS in short) of X.

Definition 2-8: [11]

Let A be a PFS in a PFTS X. A is called a Pythagorean fuzzy semi-open set (PFSOS in short) of X if $A \subseteq cl(intA)$, A Pythagorean fuzzy subset B is called a Pythagorean fuzzy semi-closed set iff $int(clB) \subseteq B$.

Definition 2-9: [6]

Let A be a PFS in a PFTS (X, τ). A is called

1) a Pythagorean fuzzy preopen set (PFPOS in short) if $A \subseteq int(cl(A))$,
2) a Pythagorean fuzzy pre-closed set (PFPCS in short) if $cl(int(A)) \subseteq A$.

Definition 2-10:

Let (X, τ) is a Pythagorean fuzzy topological space and A be a Pythagorean fuzzy subset in X. Then the fuzzy βinterior and fuzzy β-closure of A are defined by

$$\beta int\ (A) = \cup\{U; U\ is\ a \text{ Pythagorean fuzzy } \beta \text{open subset in } X\ and\ U \subseteq A\}$$

And

$$\beta cl\ (A) = \cap\{K; K\ is\ a \text{ Pythagorean fuzzy} \beta \text{ closed subset in } X\ and\ A \subseteq K\}.$$

Theorem 2-11:

Let A be a PFS in a PFTS X. A is Pythagorean fuzzy β-open set of X iff there exists a preopen set B such that $B \subseteq A \subseteq cl(B)$.

Proof:

Suppose that A is a Pythagorean fuzzy β open set. By definition, $A \subseteq cl(int(cl(A)))$, let $B = int(cl(A))$, since every Pythagorean fuzzy set of the form $int(cl(A))$ is preopen, B is a preopen set. moreover, $B \subseteq A$, and $A \subseteq cl(B)$. Hence $B \subseteq A \subseteq cl(B)$.

Conversely, suppose that for Pythagorean fuzzy set A there exists a Pythagorean fuzzy preopen set B such that $B \subseteq A \subseteq cl(B)$. Since B is Pythagorean fuzzy preopen, $B \subseteq int(cl(B))$. Taking closure on both sides gives $cl(B) \subseteq cl(int(cl(B))) \subseteq int(cl(A))$. Applying the closure operator $cl(int(cl(B))) \subseteq cl(int(cl(A)))$. Therefore $A \subseteq cl(int(cl(A)))$, which is exactly the definition of Pythagorean fuzzy β-open set. Hence, A is Pythagorean fuzzy β-open set.

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Theorem 2-12: [8]
For A Pythagorean topological space X

$$\begin{tikzcd}
 & \text{PFOS (PFCS)} \arrow[d] & \\
 & \text{PF}\alpha\text{OS (PF}\alpha\text{CS)} \arrow[dl] \arrow[dr] & \\
\text{PFPOS (PFPCS)} \arrow[dr] & & \text{PFSOS (PFSCS)} \arrow[dl] \\
 & \text{PF}\beta\text{OS (PF}\beta\text{CS)} &
\end{tikzcd}$$

Definition 2-13: [7]
Let (X, τ) and (Y, τ) be two Pythagorean fuzzy topological spaces and let $f: X \to Y$ be a function. Then, f is said to be Pythagorean fuzzy continuous if for any Pythagorean fuzzy subset A of X and for any neighborhood V of $f[A]$ there exists a neighborhood U of A such that $f[U] \subseteq V$.

Theorem 2-14: [7]
Let (X, τ) and (Y, τ) be two Pythagorean fuzzy topological spaces. A function $f: X \to Y$ is Pythagorean fuzzy continuous iff for each open Pythagorean fuzzy subset? B of Y we have $f\ [B]$ is an open Pythagorean fuzzy subset of X.

2- β-compactness in Pythagorean fuzzy topological space

Definition 3-1: [9]
A Pythagorean fuzzy topological space X is called a Pythagorean fuzzy compact space if every open cover of X has a finite subcover, that is, if for every open cover $\{U_s\}_{s\in S}$ of the space X There exists a finite set $\{s_1, s_2, \dots, s_k\} \subseteq S$ such that $X = U_{s_1} \cup U_{s_2} \cup \dots \cup U_{s_k}$.

Definition 3-2:
A Pythagorean fuzzy topological space X is called a β-compact space if every β-open the cover of X has a finite subcover.

Definition 3-3:
A Pythagorean fuzzy topological space X is called a strongly compact space if every preopen cover of X has a finite subcover.

Definition 4-3:
A Pythagorean fuzzy topological space X is called a quasi H-closed space if for every open cover $\{U_i\}_{i\in I}$, we can write $X = \bigcup_{i=1}^{n} cl(U)$.

Definition 3-5:
A Pythagorean fuzzy space X is called Extremally disconnected Pythagorean topological space(E.D) if $cl(U) \in \tau, \forall\ U \in \tau$.

Definition 3-6:
A Pythagorean topological space X is called Hausdorff space, or T_2, if for any two distinct points $x, y \in X$ there are open sets U, V such that $x \in U, y \in V, U \cap V = \emptyset$

Theorem 3-7:
If a Pythagorean fuzzy topological space X is E.D, then the closure of every Pythagorean fuzzy β-open set of X is open

Proof:
Assume that X is a Pythagorean topological space extremally disconnected, i.e., $cl(U)$ is Pythagorean fuzzy open whenever U is a Pythagorean fuzzy open set. let A be any Pythagorean fuzzy semi-preopen subset of X. By Theorem (2-11) there exists a Pythagorean fuzzy preopen set B such that $B \subseteq A \subseteq cl(B)$, since $B \subseteq A$, we have $cl(B) \subseteq cl(A)$. Also, from $A \subseteq cl(B)$, we have $cl(A) \subseteq cl\big(cl(B)\big) = cl(B)$, hence $cl(A) = cl(B)$. because B is a Pythagorean fuzzy preopen set, then $B \subseteq int(cl(B))$. Put $U = int(cl(B))$ then U *is* Pythagorean fuzzy *open and* $B \subseteq U \subseteq$

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$cl(B)$. therefore $cl(U) = cl(B)$. since U is Pythagorean fuzzy open and X is E.D., then $cl(U)$ is open. Thus, the closure of every Pythagorean fuzzy β-open set of X is open.

Theorem 3-7:

A Pythagorean fuzzy topological space X is E.D, iff for every pair of Pythagorean fuzzy open sets A, B. $cl(A \cap B) = cl(A) \cap cl(B)$.

Proof:

$\Longrightarrow$ Assume that X is a Pythagorean fuzzy extremally disconnected, and let A, B be Pythagorean fuzzy open subsets of X. since $cl(B)$ is Pythagorean fuzzy open, we use the well-known fact $[A \cap cl(B) \subseteq cl(A \cap B)]$, then we have $cl(A) \cap cl(B) \subseteq cl(cl(A) \cap B) \subseteq cl(A \cap B)$. The reverse inclusion $cl(A \cap B) \subseteq cl(A) \cap cl(B)$. Always holds by monotonicity of the closure operator. therefore $cl(A) \cap cl(B) = cl(A \cap B)$

$\Longleftarrow$ Assume that $cl(A) \cap cl(B) = cl(A \cap B)$ for every pair of Pythagorean fuzzy open subsets A, B, let A be any Pythagorean fuzzy open subset of X. To show that $cl(A)$ is Pythagorean fuzzy open, let $x \in cl(A)$. Choose any Pythagorean fuzzy open neighborhood B of x, since $x \in cl(A) \cap cl(B)$, the hypothesis gives $x \in cl(A) \cap cl(B) = cl(A \cap B)$, thus $A \cap B \neq \emptyset$, hence $x \in int\big(cl(A)\big)$, since x was arbitrary, $cl(A) \subseteq int\big(cl(A)\big) \subseteq cl(A)$, consequently $int(cl(A)) = cl(A)$, so $cl(A)$ is Pythagorean fuzzy open set, therefore X is extremally disconnected.

Lemma 3-8:

Let $A \subseteq B$ such that B is Pythagorean fuzzy α-open in X, then $A \in PF\beta O(B) \Leftrightarrow A \in PF\beta O(X)$.

Proof:

since B is α-Pythagorean fuzzy open in X , for every subset $A \subseteq B$ we have $\beta cl_B(A) = \beta cl_X(A) \cap B$.

Assume first that $A \in PF\beta O(B)$. Then $A \subseteq cl_B\Big(int_B\big(cl_B(A)\big)\Big)$

Using that hereditary property of β openness with respect to α Pythagorean fuzzy open subspaces, we obtain

$$cl_B\Big(int_B\big(cl_B(A)\big)\Big) = cl_X\Big(int_X\big(cl_X(A)\big)\Big) \cap B$$

Hence $A \subseteq cl_X\Big(int_X\big(cl_X(A)\big)\Big)$. Which shows that $A \in \beta O(X)$.

Conversely, suppose that $A \in \beta O(X)$, then $A \subseteq cl_X\Big(int_X\big(cl_X(A)\big)\Big)$. Intersecting both sides with B and using $A \subseteq B$, we get $A \subseteq cl_X\Big(int_X\big(cl_X(A)\big)\Big) \cap B$. Again , since B is α-open , $cl_X\Big(int_X\big(cl_X(A)\big)\Big) \cap B = cl_B\Big(int_B\big(cl_B(A)\big)\Big)$. Therefore $A \subseteq cl_B\Big(int_B\big(cl_B(A)\big)\Big)$, and thus $A \in PF\beta O(X)$,There fore $A \subseteq PF\beta O(B) \Leftrightarrow A \subseteq PF\beta O(X)$.

Lemma 3-9:

For a Pythagorean topological space X, the following statements are equivalent:

1- A space X is E.D
2- If $A \in PFSO(X, \tau)$ and $B \in PF\beta O(X, \tau)$ then $cl(A) \cap cl(B) = cl(A \cap B)$.
3- If $A \in PFSO(X, \tau)$ and $B \in PF\beta O(X, \tau)$ then $A \cap B \in \beta O(X, \tau)$.

Proof:

$1 \Longrightarrow 2$: let $A \in PFSO(X)$and $B \in PF\beta O(X)$ By theorem (3-7) $cl(B)$is open in X , and we obtain
$cl(A) \cap cl(B) \subseteq cl\big(int(A)\big) \cap cl(B) \subseteq cl\big(int(A) \cap cl(B)\big) \subseteq cl(A \cap B)$.
Therefore, we have $cl(A) \cap cl(B) = cl(A \cap B)$.
$2 \Longrightarrow 3$: let $A \in PFSO(X)$ and $B \in PF\beta O(X)$. Then, we have

$$A \cap B \subseteq cl(int(A)) \cap cl(int\big(cl(B)\big) = cl(int(A)) \cap int(cl(B))$$
$$= cl(int(A \cap cl(B)) \subseteq cl(int\big(cl(A)\big) \cap cl(B) = cl(int\big(cl(A \cap B)\big))$$

This shows that $A \cap B \in PF\beta O(X)$
$3 \Longrightarrow 1$: it is enough to show $cl(A) \cap cl(B) = cl(A \cap B)$ for all Pythagorean fuzzy open sets A and B
Let A and B be any Pythagorean fuzzy open sets of X. then, $cl(A)$ and $cl(B)$ are Pythagorean fuzzy semi open and hence $cl(A) \cap cl(B) \in PF\beta O(X)$. Therefore, we have

$$cl(A) \cap cl(B) \subseteq cl\big(int(cl(A) \cap B)\big) = cl(int(cl(A) \cap int\big(cl(B)\big)) \subseteq cl\Big(cl(A) \cap int\big(cl(B)\big)\Big) \subseteq cl(A \cap int(cl(B))$$
$$\subseteq cl(A \cap cl(B)) \subseteq cl(A \cap B)$$

Consequently, we obtain $cl(A) \cap cl(B) = cl(A \cap B)$.

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Lemma 3-10:
For a Pythagorean topological fuzzy space X, the following conditions are equivalent:
1- A space X is E.D
2- $PF\beta O(X) = PFPO(X)$
3- For every $A \in PF\beta O(X)$ then $cl(A) \in \tau$
4- For every $A \in PFPO(X)$ then $\beta cl(A) \in \tau$

Proof:
1⟹ 2 : Assume that X is extremally disconnected, and A is Pythagorean fuzzy β-open set, then $A \subseteq cl(int(cl(A)))$, but U is Pythagorean fuzzy open set in X ,then $cl(U)$ is Pythagorean fuzzy open and $U = int\left(cl(A)\right)$. Moreover $U \subseteq cl(A)$. hence $cl(U) \subseteq cl(A)$, since $cl(U)$ is Pythagorean fuzzy open set , we have $cl(U) \subseteq int\left(cl(A)\right)$, it follows that $A \subseteq int(cl(A))$. Therefore A is Pythagorean fuzzy preopen set i.e. $PF\beta O(X) \subseteq PFPO(X)$, but $PFPO(X) \subseteq PF\beta O(X)$ from theorem (2-12) consequently $PF\beta O(X) = PFPO(X)$.
2 ⟹ 3 : Assume that $PF\beta O(X) = PFPO(X)$. let $A \in PF\beta O(X)$. Then $A \in PFPO(X)$, since A is Pythagorean fuzzy preopen set $A \subseteq int(cl(A))$., also $int(cl(A))$ is Pythagorean fuzzy open and dance in *cl(A).* hence $cl(A) = cL(int(cl(A)))$ .Now $int(cl(A))$ is open, by equality in (2), the closure of every Pythagorean fuzzy β-open set is Pythagorean fuzzy open. therefore $cl(A) \in \tau$.
3 ⟹ 4 : Assume that $A \in PF\beta O(X)$ and $cl(A) \in \tau$, let $A \in PFPO(X)$, BY theorem (2-12) every Pythagorean fuzzy preopen set is Pythagorean fuzzy β-open, by (3) $cl(A) \in \tau$.
Since $cl(A)$ is Pythagorean fuzzy open and contains A, the β-Pythagorean fuzzy closure of A conincides with its ordinary closure. Therefore, $\beta cl(A) = cl(A)$. Hence $\beta cl(A) \in \tau$.
4 ⟹ 1 : Assume that $A \in PFPO(X)$,$\beta cl(A) \in \tau$, and let U be any arbitrary Pythagorean fuzzy open subset of X. By theorem (2-12) every Pythagorean fuzzy open set is Pythagorean fuzzy preopen, therefore, $U \in PFPO(X)$, by(4) $\beta cl(U) \in \tau$, for a Pythagorean fuzzy open set U, $\beta cl(U) = cl(U)$.Hence $cl(U) \in \tau$. Thus the closure of every Pythagorean fuzzy open set *is* Pythagorean fuzzy open. Therefore X *is* Pythagorean fuzzy extremally disconnected.
Remark 3-11: Since $\tau \subseteq PF\beta O(X,\tau)$, then every β-compact space is compact.
Theorem 3-12:
Every Pythagorean fuzzy β-closed set in a β-compact space is a β-compact set.
Proof:
let A be a Pythagorean fuzzy β-closed set in a β-compact space X, and let $\{U_i : i \in I\}$ be a Pythagorean fuzzy β-open cover of the space X. Since A is a Pythagorean fuzzy β-closed set then, X / A is Pythagorean fuzzy β-open set and the family $\{U_i : i \in I\} \cup \{X / A\}$ is a Pythagorean fuzzy β-open cover of the space X, since X is β-compact space so, $X = \bigcup_{i=1}^{n} U_i \cup (X / A)$ this means $A \subseteq \bigcup_{i=1}^{n} U_i$, so A is β-compact.
Theorem 3-13:
Let X be a Pythagorean fuzzy topological space, B *is* β-compact set and U is Pythagorean fuzzy β-open set such that $U \subseteq B$ then $B \setminus U$ is β-compact.
Proof:
Let $\{U_i : i \in I\}$ be a Pythagorean fuzzy β-open cover of $B \setminus U$ then, $\{U_i : i \in I\}$ is Pythagorean fuzzy β-open cover of the set B and since B *is* β-compact then, $B \subseteq \bigcup_{i=1}^{n} U_i \cup (U)$. But $B = (B \setminus U) \cup (U) \subseteq \bigcup_{i=1}^{n} U_i \cup (U)$ this means $(B \setminus U) \subseteq \bigcup_{i=1}^{n} U_i$ hence $B \setminus U$ is β-compact.
Theorem 3-14:
Let X be a Pythagorean fuzzy topological space, F *is* Pythagorean fuzzy β-closed set and A is β-compact set then $A \cap F$ is β-compact.
Proof:
Let $\{U_i : i \in I\}$ be Pythagorean fuzzy β-open cover of $A \cap F$ in X, this means $A \cap F \subseteq \bigcup_{i\in I} U_i$ thus $A = (A \cap F) \cup (A\setminus F) \subseteq (\bigcup_{i\in I} U_i) \cup (X\setminus F)$. since F is Pythagorean fuzzy β-closed then $X\setminus F$ is Pythagorean fuzzy β-open in X therefore $\{U_i : i \in I\} \cup \{X\setminus F\}$ is Pythagorean fuzzy β-open cover of A ,and since A is β-compact then $A = (\bigcup_{i=1}^{n} U_i) \cup (X\setminus F)$ so, $A \cap F \subseteq \bigcup_{i=1}^{n} U_i$. Thus $A \cap F$ is β-compact.

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Theorem 3-15:

Let $B \subseteq X$, where B is Pythagorean fuzzy α-open and let $A \subseteq B$,then A *is* β-compact set in $B \Leftrightarrow A$ *is* β-compact set in X.

Proof:

$\Longrightarrow$: Let $\{U_i : i \in I\}$ be Pythagorean fuzzy β-open cover of A in B that is $A \subseteq \bigcup_{i\in I} U_i$, and by lemma (3-8) for every $i \in I$ we have $U_i = V_i \cap B$ where $V_i \in PF\beta O(X), \forall\, i \in I$ therefore $\{V_i : i \in I\}$ be Pythagorean fuzzy β-open cover of A in X and since A is β-compact then $A \subseteq \bigcup_{i=1}^{n} V_i$ then $A \subseteq (\bigcup_{i=1}^{n} V_i) \cap B = \bigcup_{i=1}^{n}(V_i \cap B) = \bigcup_{i=1}^{n} U_i$ Thus A is β-compact in B.
$\Longleftarrow$: Let $\{U_i : i \in I\}$ be Pythagorean fuzzy β-open cover of A in X. Then $\{U_i \cap B : i \in I\}$ be Pythagorean fuzzy β-open cover of A in B, but B is Pythagorean fuzzy α-open in X and β-compact in B , then $A \subseteq \bigcup_{i=1}^{n}(U_i \cap B) = \bigcup_{i=1}^{n} U_i$. Thus A is β-compact in X.

Theorem 3-16:

Let$(X, \tau_1), (Y, \tau_2)$be two Pythagorean fuzzy topological spaces and $f: X \to Y$ a Pythagorean fuzzy continuous surjection. If (X, τ_1) is Pythagorean fuzzy β compact, then , (Y, τ_2) is a Pythagorean fuzzy compact.

Proof. Let $\{U_i : i \in I\}$ be Pythagorean fuzzy β-open cover of Y . Then by Theorem (2.14) $\{f^{-1}(U_i) : i \in I\}$is a Pythagorean fuzzy open cover of X, too. Since X is Pythagorean fuzzy βcompact, then there exists a finite subfamily $\{U_i : i = 1,2, \dots, n\}$such that $\cup f^{-1}(U_i) = 1_X$. From the subjectivity of f we obtain$(\bigcup_{i=1}^{n} f^{-1}(U_i)\,) \subseteq \bigcup_{i=1}^{n} f(f^{-1}(U_i)) = \bigcup_{i=1}^{n} U_i \; = 1_Y$. Hence (Y, τ_2) is Pythagorean fuzzy β-compact.

Theorem 3-17:

If A is a β-compact subset of a Pythagorean Hausdorff space X and $x \notin A$, then there exist two Pythagorean fuzzy open sets U, V such that $x \in U$, $A \subseteq V$ and $U \cap V = \emptyset$.

Proof:

Let X be a Pythagorean fuzzy Hausdorff space, A is Pythagorean fuzzy β-compact set. For any two points x, a where $x \in X\backslash A$ and $a \in A$ there exist two Pythagorean fuzzy open sets U_a, V_x such that $a \in U_a$ and $x \in V_x$, $U_a \cap V_x = \emptyset$. It is clear that $\{U_{a_i}, a \in A, i \in I\}$ Pythagorean fuzzy open cover of the Pythagorean fuzzy set A and since A is a β-compact then $A \subseteq \bigcup_{i=1}^{n} U_{a\,i}$.

Let $U = \bigcup_{i=1}^{n} U_{a\,i}$, then U is Pythagorean fuzzy open. On the other hand, we have $x \in V_{x_i}$,$i \in I$ then $\bigcap_{i=1}^{n} V_{x_i} = V$ is Pythagorean fuzzy open and since $V_{x_i} \cap U_{a_i} = \emptyset, \forall\, i \in I$ then $U \cap V = \emptyset$ where $x \in U$, $A \subseteq V$.

Lemma 3-18:

Let X be Pythagorean fuzzy Hausdorff space, then $PF\beta O(X) = PFPO(X)$.

Proof:

Let $A \in PF\beta O(X)$, $x \in X\backslash int(cl(A))$ and we take $y \in int(cl(A))$. Since $x \neq y$ and X is a T_2 space, there exist two Pythagorean open sets U, V such that $x \in U$and $y \in V$,$U \cap V = \emptyset$. Let $V = int(cl(A)$ then $U \cap int(cl(A) = \emptyset, \forall\, x \in X\backslash int(cl(A))$ therefore $x \notin cl\left(int(cl(A))\right)$, $\forall x \in X\backslash int(cl(A))$ and so, $x \in X\backslash cl(int(cl(A))) \Rightarrow X\backslash int(cl(A)) \subseteq X\backslash cl\left(int(cl(A))\right) \Longrightarrow cl\left(int(cl(A))\right) \subseteq int(cl(A))$. Since $A \in PF\beta O(X)$,then $A \subseteq int(cl(A) \Leftrightarrow A \in PFPO(X)$ so, $PF\beta O(X) \subseteq PFPO(X)$ and we have always $PFPO(X) \subseteq PF\beta O(X)$, thus $PF\beta O(X) = PFPO(X)$.

Remark 3-19:

Every Pythagorean fuzzy T_2 Strongly compact space is β-compact space

Lemma 3-20:

Let (X, τ) be a Pythagorean fuzzy topological space satisfies the following property: Every Pythagorean fuzzy β-open set is dens in this space then $PF\beta O(X) = PFPO(X)$

Proof:

Let $A \in PF\beta O(X)$ be dens in (X, τ). Suppose that A is not Pythagorean fuzzy open then, $A \nsubseteq int(cl(A)$ this means $A\backslash int(cl(A) \neq \emptyset$, that is $\exists\, x \in A : x \notin int(cl(A) \subseteq cl(A)$, $\forall\, U \in \tau\backslash\{\emptyset\}$. Since $x \in U$ then, $U \not\subset cl(A)$ and so, either $cl(A) \subseteq U$ or $U \cap cl(A) = \emptyset$. If $cl(A) \subseteq U$ this is a contradiction with the dens hypothecs.

If $U \cap cl(A) = \emptyset$ then, $U \cap A = \emptyset$ also this is a contradiction with the dens hypothecs. Thus $A \in PFPO(X)$ and so, $PF\beta O(X) \subseteq PFPO(X)$ and we have always $PFPO(X) \subseteq PF\beta O(X)$. Hence $PF\beta O(X) = PFPO(X)$.

Remark 3-21:

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If X is a Pythagorean fuzzy Strongly compact space such that every Pythagorean fuzzy β-open set in X is dens, then X is β-compact.

Theorem 3-22:

Every β-compact set in Pythagorean fuzzy Hausdorff space is Pythagorean fuzzy β-closed.

Proof:

Let X be a Pythagorean fuzzy Hausdorff space, A is β-compact subset in X. By theorem (3-17) for any $x \notin A$ there exist two Pythagorean fuzzy open sets U, V such that $x \in U$, $A \subseteq V$ and $U \cap V = \emptyset$. Since $A \subseteq V$, then $U \cap A = \emptyset$ so, $x \notin cl(A)$ but $\beta cl(A) \subseteq cl(A) \Longrightarrow x \notin \beta cl(A)$. Thus $\beta cl(A) \subseteq A$ and hence, A is Pythagorean fuzzy β-closed.

Theorem 3-23:

If X is Pythagorean fuzzy E.D and Strongly compact space then, X is β-compact space.

Proof:

Let $\{U_i : i \in I\}$ be Pythagorean fuzzy β-open the cover of X.By Lemma (3-10) the last cover is a Pythagorean fuzzy preopen cover of X with a finite subcover (because X is a Pythagorean fuzzy strongly compact space). Hence X is β-compact space.

Theorem 3-24:

If X is Pythagorean fuzzy E.D and β-compact space then X is a quasi-H-closed space.

Proof:

Let $\{U_i : i \in I\}$ be a Pythagorean fuzzy open cover of X. Since X is Pythagorean fuzzy E.D space then, $cl(U_i) \in \tau, \forall i \in I$ The family $\{cl(U_i) : i \in I\}$ is both a Pythagorean fuzzy open cover and a Pythagorean fuzzy β-open the cover of X. Since X is β-compact then $X = \bigcup_{i=1}^{n} cl(U_i)$. Hence X is quasi H-closed space.

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