Alq J Sci. 2026;2(7):699-703
https://doi.org/10.69667/ajs.26703

Alqalam Journal of Science
مجلـــة القلم للعلـوم
https://alqalam.utripoli.edu.ly/index.php/AR

Original article

Solving Two –Dimensional Volterra Integral Equation Using the Double Laplace Transform

Mona Sanosi

Department of Statistics, Faculty of Education, Abi Issa, Zawia, Libya
Email. mo.salamah@zu.edu.ly

Abstract
This study investigates an efficient analytical approach for solving two-dimensional Volterra integral equations by employing the double Laplace transform technique. To establish a solid mathematical foundation, we first review the essential definitions, core concepts, and fundamental theoretical properties governing the double Laplace transform. Particular emphasis is placed on the convolution theorem and its practical implications, alongside a rigorous discussion on the existence and uniqueness of the solution. To demonstrate the validity and effectiveness of the discussed framework, the method is successfully implemented to resolve specific instances of Volterra integral equations, showcasing its accuracy and computational reliability.
Keywords. Two-Dimensional Volterra Integral Equations, Double Laplace Transform, Convolution Property, Analytical Solutions, Existence and Uniqueness Theorem.

Introduction

Integral transforms serve as an indispensable mathematical framework for resolving complex differential and integral equations. The foundational concept of the double Laplace transform for functions of two variables dates back to the pioneering work of Bernstein in 1939[1]. Since its introduction, this transform has evolved into a powerful analytical apparatus utilized extensively in analyzing functional, integral, and partial differential equations [2]. Over the decades, numerous investigations have established its core theoretical properties, including linearity, differentiation rules, and the vital convolution theorem [3-7]. These advancements have firmly positioned the double Laplace transform as a robust tool for tackling intricate integro-differential problems often encountered across engineering and applied sciences. The primary advantage of employing integral transforms lies in their capacity to transmute sophisticated integral operations into simplified algebraic or differential structures, thereby paving a direct pathway to exact solutions.

Crucially, this methodology bypasses the need for restrictive approximations such as linearization or numerical discretization. While a single integral transform can reduce integral equations to ordinary differential equations, utilizing a double transform—such as Laplace, Fourier, Elzaki, Natural, Sumudu, or ARA transforms—effectively reduces them to straightforward algebraic equations. Specifically, the double Laplace transform generalizes the classical one-dimensional variant by mapping integration and convolution operations directly into algebraic multiplication. Consequently, the transformed equation can be solved using basic algebra before recovering the final exact solution via the inverse transform [8,9].

In this paper, we leverage the utility of the double Laplace transform, aided by the convolution theorem, to evaluate and solve specific classes of two-dimensional Volterra integral equations. To ensure a coherent development of ideas, the study is structured systematically. We begin by introducing the essential definitions and fundamental properties that constitute the theoretical foundation of the double Laplace transform. Next, a rigorous assessment of the existence and uniqueness of the solution is provided to guarantee the mathematical validity of the approach. The pivotal role of the convolution property in simplifying integral operations is then thoroughly examined. Finally, the practical efficacy and computational reliability of this framework are demonstrated through a series of illustrative examples, bridging the gap between theoretical formulation and practical implementation.

Basic concepts

Definition (1) [10]***: Laplace transform***

Let q(x) be a piecewise continuous function on the interval [0,∞]. The Laplace transform of $q(x)$, denoted by$\mathcal{L}\,\{q(x)\}$ or $Q(s)$and known as

Received*: 02-05-2026 -* ***Accepted****: 01-07-2026 -* ***Published****: 09-07-2026*

Alq J Sci. 2026;2(7):699-703
https://doi.org/10.69667/ajs.26703
Alqalam Journal of Science
مجلـــة القلم للعلوم
https://alqalam.utripoli.edu.ly/index.php/AR

$\mathcal{L}\{q(x)\} = Q(s) = \int_0^{\infty} e^{-sx} q(x)dx$,……..(1)
where s is a complex number.

Definition *(2):*
The inverse Laplace transform is defined by

$$\mathcal{L}^{-1}\{Q(s)\} = q(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{sx} q(s)ds, \ldots..(2)$$

where c is a real constant.

Definition (3) [10] *:*
The function $\delta(x)$ which is zero everywhere except at $x = 0$, and tends to infinity in such a manner that

$$\int_{-\infty}^{+\infty} \delta(x)dx = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(3)$$

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$

Definition *(4):*
Double Laplace transform of a function of two variables $q(x, y)$for $x, y > 0$ is denoted by
$\mathcal{L}_2\{q(x, y)\} = \bar{\bar{q}}(s, p) = \int_0^{\infty}\int_0^{\infty} e^{-sx-py} q(x, y)dxdy$, where $x, y \in \mathbb{R}$ …….(4)

Definition (5):
The inverse double Laplace transform $\mathcal{L}_2^{-1}\{\bar{\bar{q}}(s, p)\} = q(x, y)$is defined by the complex double integral formula $\mathcal{L}_2^{-1}\{\bar{\bar{q}}(s, p)\} = q(x, y) = \frac{1}{(2\pi i)^2}\int_{c-i\infty}^{c+i\infty}\int_{d-i\infty}^{d+i\infty} e^{sx+py}\bar{\bar{q}}(s, p)ds\, dp \ldots\ldots\ldots\ldots\ldots..(5)$

Definition (6):
The double integral equation of the form
$q(x, y) = h(x, y) + \lambda \int_0^x \int_0^y q(x - \tau, y - \sigma)g(\tau, \sigma)d\tau d\sigma$,……….(6)
where q is an unknown function, λ is a given constant parameter, $q(x, y)$and $g(\tau, \sigma)$are known functions.

Definition (7):
The function $\delta(x, y)$ which is zero everywhere except at
$(x, y) = (0,0)$, and tends to infinity in such a manner that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \delta(x, y)dxdy = 1 \ldots\ldots\ldots\ \ \ldots\ldots.(7)$$

$$\delta(x, y) = \begin{cases} 0 & x \neq 0, y \neq 0 \\ \infty & x = 0, y = 0 \end{cases}$$

In two dimensions, for a function$q(x, y)$, we have that,
$\iint \delta(x - a)q(y - b)dxdy = q(x, y) \ldots\ldots\ldots\ldots\ldots(8)$
Where $\delta(x - a , y - b)$ Is a δ-function located at position? a, b

Definition (8):
Let $q(x, y)$and $g(x, y)$ be continuous functions on. Double convolution of the functions $q(x, y)$ and $g(x, y)$ exists and is defined by:

$$(q ** g)(x, y) = \int_0^x \int_0^y q(x - \tau, y - \sigma)\, g(\tau, \sigma)\, d\tau d\sigma \ldots\ldots(9)$$

The convolution is commutative, that is,
$(q ** g)(x, y) = (g ** q)(x, y)$.

Theorem 1.1 (Convolution Theorem).
If $\mathcal{L}_2\{q(x, y)\} = \bar{\bar{q}}(s, p)$ and $\mathcal{L}_2\{g(x, y)\} = \bar{\bar{g}}(s, p)$,
then

$$\mathcal{L}_2\{(q ** g)(x, y)\} = \mathcal{L}_2\{q(x, y)\}\mathcal{L}_2\{g(x, y)\} = \bar{\bar{q}}(s, p)\bar{\bar{g}}(s, p) \ldots\ldots\ldots\ldots(10)$$

Received*: 02-05-2026 -* ***Accepted****: 01-07-2026 -* ***Published****: 09-07-2026*

Alq J Sci. 2026;2(7):699-703
https://doi.org/10.69667/ajs.26703

Alqalam Journal of Science
مجلــة القلم للعلـوم
https://alqalam.utripoli.edu.ly/index.php/AR

Or, equivalently,

$$\mathcal{L}_2^{-1}\{\bar{\bar{q}}(s,p)\bar{\bar{g}}(s,p)\} = (q ** g)(x,y) \dots\dots\dots\dots\dots..(11)$$

where $(q ** g)(x,y)$ is defined by the double integral (9)

Theorem 1.2. [Linearity of the double Laplace transform]

Let $q(x,y)$ and $g(x,y)$ be functions whose double Laplace transform and $k_1$are k_2 constants. Then,

$$\mathcal{L}_2[k_1 q(x,y) + k_2 g(x,y)] = k_1\bar{\bar{q}}(s,p) + k_2\bar{\bar{g}}(s,p) \dots (12)$$

Theorem 1.3: [Linearity of the inverse double Laplace transform]

Assume $\bar{\bar{q}}(s,p), \bar{\bar{g}}(s,p)$ that exists and c_1 , c_2 are continuous. Then,

$$\mathcal{L}^{-1}{}_2[c_1\bar{\bar{q}}(s,p) + c_2\bar{\bar{g}}(s,p)] = c_1 q(x,y) + c_2 g(x,y) \dots\dots\dots (13)$$

Existence of double Laplace transforms:

Let $q(x,y)$ is an exponential order $n_1$and n_2 $as\ x,y\ to\ \infty$.

If $\exists\, K > 0$, such that $\forall x > X, y > Y$, we have

$|q(x,y)| \le K e^{n_1 x + n_2 y}$….(14)

We can write $q(x,y) = O(e^{n_1 x + n_2 y})$ $as\ x\ and\ y\ goes\ to\ \infty$.

Theorem 1.4.

If a function $q(x,y)$ is a continuous function defined on $(0,X)$ and $(0,Y)$ with exponential order $e^{n_1 x + n_2 y}$, then the double Laplace transform of $q(x,y)$ exist for all $s$and $p$provided $Re(s) < n_1$, $Re(p) < n_2$

Proof: By placing Equation (14) into Equation (4), we obtain

$$|\bar{\bar{q}}(s,p)| = |\mathcal{L}_2\{q(x,y)\}| =$$

$$\left|\int_0^\infty\int_0^\infty e^{-sx-py} q(x,y)dxdy\right| = \int_0^\infty\int_0^\infty e^{-sx-py}|q(x,y)|dxdy \le N\int_0^\infty\int_0^\infty e^{-(s-n_1)x}\, e^{-(p-n_2)y}\, dxdy = \frac{N}{(s-n_1)(p-n_2)},\ \ Re(s) > n_1\, , Re(p) > n_2$$

It follows that

$$\lim_{\substack{x\to\infty \\ y\to\infty}} |\bar{\bar{q}}(s,p)| = 0\ \ or\ \ \lim_{\substack{x\to\infty \\ y\to\infty}} \bar{\bar{q}}(s,p) = 0\,.$$

Table (1). The Double Laplace Transform of Some Elementary Functions

S.N	$\boldsymbol{q(x,y)}$	$\boldsymbol{\mathcal{L}_2\{q(x,y)\} = q(s,p)}$
1	1	$\frac{1}{sp}$
2	e^{ax+by}	$\frac{1}{(s-a)(p-b)}$
3	$\cos(ax+by)$	$\frac{sp-ab}{(s^2+a^2)(p^2+b^2)}$
4	$\sin(ax+by)$	$\frac{ap+bs}{(s^2+a^2)(p^2+b^2)}$
5	$(xy)^n$	$\frac{(n!)^2}{(sp)^{n+1}}$
6	$x^m y^n$	$\frac{m!}{s^{m+1}}\frac{n!}{p^{n+1}}$,m,n $is\ positive$
7	$J_0 a\sqrt{xy}$	$\frac{4}{4sp+p^2}$
8	$\frac{1}{\sqrt{xy}}$	$\frac{\pi}{\sqrt{sp}}$
9	$e^{ax+by}\ g(x,y)$	$\bar{\bar{g}}(x+a,y+b)$
10	$\delta(x,y)$	1

Received*: 02-05-2026 -* ***Accepted****: 01-07-2026 -* ***Published****: 09-07-2026*

Alq J Sci. 2026;2(7):699-703
https://doi.org/10.69667/ajs.26703

Alqalam Journal of Science
مجلـــة القلم للعلـوم
https://alqalam.utripoli.edu.ly/index.php/AR

Solution of Integral Equations by the Double Laplace Transform

To illustrate the basic idea of this method for solving two-dimensional Volterra integral equations, we consider the general form

$$q(x,y) = h(x,y) + \lambda \int_0^x \int_0^y q(x-\xi, y-\eta) g(\xi,\eta) d\xi d\eta \ldots (15)$$

Where $q(.,.)$ is the unknown function, λ is a given constant parameter, $h(x,y)$ and $g(x,y)$ Are known functions.
Now, applying the double Laplace transform on both sides of Equation (15), we find that $\mathcal{L}_2\{q(x,y)\} = \bar{\bar{q}}(s,p)$defined by (4) so that the convolution integral equation reduces to the form

$$\bar{\bar{q}}(s,p) = \bar{\bar{h}}(s,p) + \lambda \mathcal{L}_2\{(q ** g)(x,y)\}$$

which is, by the convolution Theorem 1.1,

$$\bar{\bar{q}}(s,p) = \bar{\bar{h}}(s,p) + \lambda \bar{\bar{q}}(s,p) \bar{\bar{g}}(s,p)$$
$$\bar{\bar{q}}(s,p) - \lambda \bar{\bar{q}}(s,p) \bar{\bar{g}}(s,p) = \bar{\bar{h}}(s,p)$$

$\bar{\bar{q}}(s,p)\left(1 - \lambda \bar{\bar{q}}(s,p)\right) = \bar{\bar{h}}(s,p)$.......(16)

Consequently,

$\bar{\bar{q}}(s,p) = \frac{\bar{\bar{h}}(s,p)}{1-\lambda\, q(s,p)}$......................(17)

Using the inverse transform of the double Laplace transform, we get the exact solution of (16)

$$\mathcal{L}^{-1}{}_2[\bar{\bar{q}}(s,p)] = \mathcal{L}^{-1}{}_2\left[\frac{\bar{\bar{h}}(s,p)}{1-\lambda \bar{\bar{q}}(s,p)}\right]$$
$$\mathcal{L}^{-1}{}_2[\bar{\bar{q}}(s,p)] = \mathcal{L}^{-1}{}_2\left[\bar{\bar{h}}(s,p) \bar{\bar{d}}(s,p)\right]$$
$$q(x,y) = \int_0^x \int_0^y h(x-\xi, y-\eta) d(\xi,\eta) d\xi d\eta \ldots\ldots\ldots (18)$$

Where $\bar{\bar{d}}(s,p) = \frac{1}{1-\lambda \bar{\bar{q}}(s,p)}$

Thus, we obtain the formal solution of the original integral equation (15).
We illustrate the above method by simple examples.

Examples (a):

Solve

$q(x,y) = a - \lambda \int_0^x \int_0^y q(\xi,\eta) d\xi d\eta \ldots\ldots (19)$

Applying the Laplace transform on both sides of Equation (19)

$$\bar{\bar{q}}(s,p) = \mathcal{L}_2\{a\} - \lambda\, \mathcal{L}_2\{(q ** 1)(x,y)\}$$

which is, by the convolution Theorem 1.1,

$$\bar{\bar{q}}(s,p) = \frac{a}{sp} - \lambda\, \bar{\bar{q}}(s,p) \frac{1}{sp}$$
$$\bar{\bar{q}}(s,p) + \lambda\, \bar{\bar{q}}(s,p) \frac{1}{sp} = \frac{a}{sp}$$
$$\bar{\bar{q}}(s,p)\left(1 + \frac{\lambda}{sp}\right) = \frac{a}{sp}$$

As a result,

$$\bar{\bar{q}}(s,p) = \frac{a}{sp+\lambda} \ldots\ldots\ldots\ldots\ldots (20)$$

Using the inverse double Laplace transform, we get the exact solution of (20)

$$\mathcal{L}^{-1}{}_2[\bar{\bar{q}}(s,p)] = \mathcal{L}^{-1}{}_2\left[\frac{a}{sp+\lambda}\right]$$
$$q(x,y) = a\mathcal{L}^{-1}{}_2\left[\frac{1}{sp+\lambda}\right]$$

Then the solution to Equation (19) is $q(x,y) = aJ_0\left(2\sqrt{\lambda xy}\right) \ldots\ldots\ldots\ldots\ldots\ldots (21)$

***Received**: 02-05-2026 - **Accepted**: 01-07-2026 - **Published**: 09-07-2026*

Alq J Sci. 2026;2(7):699-703
https://doi.org/10.69667/ajs.26703
Alqalam Journal of Science
مجلـــة القلم للعلـوم
https://alqalam.utripoli.edu.ly/index.php/AR

Examples (b):
Solve the equation

$$\int_0^x\int_0^y q(x-\xi,y-\eta)q(\xi,\eta)d\xi d\eta \ = a^2 \ldots\ldots(22)$$

Where a is a constant. Applying the double Laplace transform of equation (22), we get

$$\overline{\overline{q^2}}(s,p)=\frac{a^2}{sp}$$

or

$$\bar{\bar{q}}(s,p)=\frac{a}{\sqrt{sp}}\ldots\ldots(23)$$

Taking $\mathcal{L}^{-1}{}_2$ For equation (23), we obtain the solution $q(x,y)$ of equation (21).

$$q(x,y)=a\,\mathcal{L}^{-1}{}_2\left[\frac{1}{\sqrt{sp}}\right]=\frac{a}{\pi}\frac{1}{\sqrt{sp}}\ldots\ldots(24)$$

Examples (c):
Solve

$$\int_0^x\int_0^y e^{x-}\mathrm{q}(x,y)dxdy=e^{x+y}+xye^{x+y}\ldots\ldots\ldots\ldots\ldots\ldots(25)$$

$$\bar{\bar{\mathrm{q}}}(s,p)\frac{1}{(s-1)(p-1)}=\frac{1}{(s-1)(p-1)}+\frac{1}{(s-1)^2(p-1)^2}$$

$$\bar{\bar{\mathrm{q}}}(s,p)=\frac{1}{(s-1)(p-1)}+1\ldots\ldots\ldots\ldots\ldots\ldots(26)$$

Using the inverse transform $\mathcal{L}_2^{-1}\{\bar{\bar{\mathrm{q}}}(s,p)\}$, we get the exact solution of equation (25)

$$\mathrm{q}(x,y)=e^{x+y}+\delta(x,y)\ldots\ldots(27)$$

Conclusions

In this study, the two-dimensional Laplace transform was successfully applied to solve two-dimensional Volterra integral equations. By using its fundamental properties and the two-dimensional convolution theorem, integral equations were transformed into algebraic equations, yielding accurate solutions efficiently without approximation techniques. This method shows strong potential for solving more complex integral and fractional differential equations in physics and engineering.

References

1. Berstein DL. The double Laplace integral [dissertation]. Providence (RI): Brown University; 1939.
2. Debnath L. The double Laplace transforms and their properties with applications to functional, integral and partial differential equations. Int J Appl Comput Math. 2016;2(2):223–41.
3. Belgacem FBM, Silambarasan R. Theory of the double Laplace transform and its applications. Int J Appl Math Stat. 2013.
4. Debnath L, Bhatta D. Integral transforms and their applications. Boca Raton (FL): CRC Press; 2015.
5. Schiff JL. The Laplace transform: theory and applications. New York: Springer; 1999.
6. Singh J, Kumar D, Sushila. Double Laplace transform and its applications to partial differential equations. Appl Math Comput. 2012.
7. Patil DP. Solution of wave equation by double Laplace and double Sumudu transform. Spec Issue IVCIMS. 2021;135–8.
8. Agarwal RP, Mofarreh F, Shah R, Luangboon W, Nonlaopon K. An analytical technique based on natural transform to solve fractional-order parabolic equations. Entropy (Basel). 2021;23(8):1086.
9. Widder DV. The Laplace transform. Princeton (NJ): Princeton University Press; 2015.
10. Dass HK. Mathematical physics: for the students of B.Sc. (Honours) and M.Sc. (Physics). New Delhi: S. Chand Publishing.

Received: *02-05-2026 -* ***Accepted***: *01-07-2026 -* ***Published***: *09-07-2026*