Alq J Sci. 2026;2(4):474-481
https://doi.org/10.69667/ajs.26411

Alqalam Journal of Science

مجلـــة القلم للعلـوم

https://alqalam.utripoli.edu.ly/index.php/AR

Original article

α-Separation Axioms in Pythagorean Neutrosophic Topological Spaces

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Abstract
The purpose of this paper is to introduce and study the notion of Pythagorean Neutrosophic α -separation axioms (α -T_i spaces, for $i = 0,1,2$). As well, several of their properties are established. In addition, we define the concepts of Pythagorean Neutrosophic α -regular and α -normal spaces.

Keywords: Neutrosophic set, Pythagorean fuzzy subsets, Pythagorean Neutrosophic topological space.

Introduction

In 1965, Zadeh [1] introduced the concept of a fuzzy set. In 1968, Chang [2] introduced the notion of fuzzy topological spaces, establishing some fundamental concepts in general topology, in 1986, Atanassov [3,4] proposed the intuitionistic fuzzy set as a generalized version of a fuzzy set. A decade later, Florentin Smarandache [5] developed and investigated the intuitionistic fuzzy set by incorporating a new philosophical framework. Subsequently, the concept of the Neutrosophic topological space was defined by Salama and Alblowi [6]. In 2013, [7,8] Yager introduced the Pythagorean fuzzy set based on Pythagorean membership grades. Following this, researchers such as Shena and Nirmala [9] explored properties of Pythagorean Neutrosophic open sets, α - open sets, and other related notions, further enriching the theoretical framework of these concepts. separation axioms have been extensively studied in neutrosophic topological spaces [10, 11,12,13], but their formulation in the framework of Pythagorean neutrosophic topology has not been sufficiently explored. In this paper, we employ the notion of Pythagorean Neutrosophic $\alpha -$ open set to introduce and investigate the concept of α - separation axioms in Neutrosophic Fuzzy Topological Space. Besides presenting some of their fundamental properties, we also define the concepts of Pythagorean Neutrosophic α-regular and Pythagorean Neutrosophic α-normal space. Moreover, several related results are established and proved.

1. Preliminaries

Definition 2.1: [7] A Pythagorean Fuzzy Subset A of a non-empty set X is a pair $A = (\mu_A, \vartheta_A)$ of a membership function $\mu_A: X \to [0,1]$ and a non-membership function $\vartheta_A: X \to [0,1], 0 \le \mu_A^2(x) + \vartheta_A^2(x) \le 1$ for each element $x \in X$.

Supposing $\mu_A^2(x) + \vartheta_A^2(x) \le 1$, then there is a degree of indeterminacy of $x \in X$ to A defined by $\pi_A(A) = \sqrt{1 - [\mu_A^2(x) + \vartheta_A^2(x)]}$ and $\pi_A(x) \in [0,1]$. We denote the set of all Pythagorean fuzzy subsets over X by $PFS(X)$.

Definition 2.2: [9] Let X be a non-empty set. Then A is said to be a Neutrosophic set (NS) of X if it is an object having the form $A = \{\langle x, \mu_A, \sigma_A, \gamma_A \rangle : x \in X\}$, were the function $\mu_A: X \to [0,1], \sigma_A: X \to [0,1]$ and $\gamma_A: X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$), degree of indeterminacy (namely $\sigma_A(x)$) and degree of non-membership (namely $\gamma_A(x)$) of each element $x \in X$ to the set A, and satisfies the condition, $0 \le \mu_A(x) + \sigma_A(x) + \gamma_A(x) \le 3$ for each $x \in X$.

Definition 2.3: [14] Let A and B be NSs of the form
$A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x) \rangle : x \in X\}$, and $B = \{\langle x, \mu_B(x), \sigma_B(x), \gamma_B(x) \rangle : x \in X\}$. Then

1) $A \subseteq B \Leftrightarrow \mu_A(x) \le \mu_B(x), \sigma_A(x) \ge \sigma_B(x)$ and $\gamma_A(x) \ge \gamma_B(x)$.
2) $A^c = \{\langle x, \sigma_A(x), \mu_A(x), \gamma_A(x) \rangle : x \in X\}$.
3) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \vee \sigma_B(x), \gamma_A(x) \vee \gamma_B(x) \rangle : x \in X\}$.
4) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \gamma_A(x) \wedge \gamma_B(x) \rangle : x \in X\}$.
5) $0 = \{\langle x, 0,1,1 \rangle : x \in X\}$, and $1 = \{\langle x, 1,0,0 \rangle : x \in X\}$.

***Received**: 18-02-2026 - **Accepted**: 15-04-2026 - **Published**: 22-04-2026*

Definition 2.4: [14] Let f be a mapping from an ordinary set X into an ordinary set Y. If $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$ is a NS in Y, then the inverse image of A under f is an NS defined by $f^{-1}(A) = \{\langle x, f^{-1}\mu_A(x), f^{-1}\sigma_A(x), f^{-1}\gamma_A(x)\rangle : x \in X\}$.

The image of NS $B = \{\langle y, \mu_B(y), \sigma_B(y), \gamma_B(y)\rangle : y \in Y\}$ under f is a NS defined by
$f(B) = \{\langle y, f\mu_B(y), f\sigma_B(y), f\gamma_B(y)\rangle : y \in Y\}$ where

$$f(\mu_B)(y) = \begin{cases} sup_{x \in f^{-1}(y)} \mu_B(x), & if \quad f^{-1}(y) \neq 0 \\ 0, & otherwise \end{cases}$$

$$f(\sigma_B)(y) = \begin{cases} inf_{x \in f^{-1}(y)} \sigma_B(x), & if \quad f^{-1}(y) \neq 0 \\ 0, & otherwise \end{cases}$$

$$f(\gamma_B)(y) = \begin{cases} inf_{x \in f^{-1}(y)} \gamma_B(x), & if \quad f^{-1}(y) \neq 0 \\ 0, & otherwise \end{cases}$$

For each $y \in Y$.

Definition 2.5: [15] Let X be a non-empty set. Then A is said to be a Pythagorean Neutrosophic set (PN) of X there is a $A = \{\langle x, \mu_A, \sigma_A, \gamma_A\rangle : x \in X\}$ where the function $\mu_A : X \to [0,1], \sigma_A : X \to [0,1]$, and $\gamma_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$), degree of indeterminacy (namely$\sigma_A(x)$ and degree of non-membership (namely $\gamma_A(x)$) of each element $x \in X$ to the set A, and satisfies the condition $0 \leq \mu_A(x)^2 + \sigma_A(x)^2 + \gamma_A(x)^2 \leq 2$.

Definition 2.6: [15] A Pythagorean Neutrosophic Topology (PNT) on a non-empty fixed set X is a family of τ of Pythagorean Neutrosophic sets in X satisfying the following conditions:

i. $0,1 \in \tau$.
ii. $\forall\, A_1, A_2 \in \tau$, we have $A_1 \wedge A_2 \in \tau$.
iii. $\cup_{i \in I} A_i \in \tau$ for any arbitrary family $\{A_i; i \in I\} \subseteq \tau$.

In this case the pair (X, τ) is called a Pythagorean Neutrosophic Topological space (PNTS) and each Pythagorean Neutrosophic subsets in τ is known as a Pythagorean Neutrosophic open set (PNOS) in X. The complement of an open Pythagorean Neutrosophic subsets is called a closed Pythagorean Neutrosophic subsets.

Definition 2.7:[15] Let (X, τ) be a PNTS and A be a PNS over X

1) Pythagorean Neutrosophic interior of A (PNint(A)) is the union of all Pythagorean Neutrosophic open sets of X contained in A. That is,
   $\text{PNint}(A) = \cup\, \{G : G \text{ is a PNO set in } X \text{ and } G \subseteq A\}$.
2) Pythagorean Neutrosophic closure of A (PNcl(A)) is the intersection of all Pythagorean Neutrosophic closed sets of X containing A. That is
   $\text{PNcl}(A) = \cap\{H : H \text{ is a PNC set in } X \text{ and } A \subseteq H\}$.

Definition 2.8: [16] Let X be a non-empty set. If a, b, c are real standard or non-standard subset of [0,1], then the Pythagorean Neutrosophic set $x_{a,b,c}$ is said to be Pythagorean Neutrosophic point (PNP) in X and it is given by:

$$x_{a,b,c}(x_p) = \begin{cases} (a,b,c) & if \quad x = x_p \\ (0,0,1) & if \quad x \neq x_p \end{cases}$$

For each $x_p \in X$ is said to be the support of $x_{a,b,c}$ where a denotes the degree of membership value, b the degree of indeterminacy and c is the degree of nonmember ship value of $x_{a,b,c}$.

Definition 2.9: [7, 14] For Pythagorean Neutrosophic set A in a Pythagorean Neutrosophic Topological Space (X, τ) is called Pythagorean Neutrosophic Fuzzy α-open set (PN α OS) if $A \subseteq PNint\left(PNcl\left(PNint(A)\right)\right)$ Pythagorean Neutrosophic set is called Pythagorean Neutrosophic α-closed (resp. Pythagorean Neutrosophic α closed.

Definition 2.10: [17] Let (X, τ) be a PNTS and A be a PNS over X

1- Pythagorean Neutrosophic α-interior of A ($PN\alpha int(A)$) is the union of all Pythagorean Neutrosophic α -open sets of X contained in A. That is, $PN\alpha int(A) = \cup\, \{G : G \text{ is a } PN\alpha O \text{ set in } X \text{ and } G \subseteq A\}$.
2- Pythagorean Neutrosophic α -closure of A (PNαcl(A)) is the intersection of all Pythagorean neutrosophic α -closed sets of X containing A. That is, $PN\alpha cl(A) = \cap\, \{H : H \text{ is a } PN\alpha C \text{ set in } X \text{ and } A \subseteq H\}$.

Remark 2.11: [9] For a Pythagorean Neutrosophic Fuzzy Topologcal Space (X, τ) and $A, B \subseteq X$. From definition We have

Received: *18-02-2026* - ***Accepted***: *15-04-2026* - ***Published***: *22-04-2026*

1- Every Pythagorean Neutrosophic open set is Pythagorean Neutrosophic Fuzzy α-open set.
2- If $A \subseteq B$, then $PN\alpha int(A) \subseteq PN\alpha int(B)$.
3- If $A \subseteq B$, then $PN\alpha cl(A) \subseteq PN\alpha cl(B)$.

Definition 2-12: [7, 9, 16]

Let f be a function from a Pythagorean Neutrosophic topological space (X, τ) to a Pythagorean Neutrosophic topological space (Y, θ) then f is called

1- A Pythagorean Neutrosophic αopen function if $f(A)$ is a Pythagorean Neutrosophic α -open set in Y for every Pythagorean neutrosophic open set A in X.
2- A Pythagorean Neutrosophic α closed function if $f(A)$ is a Pythagorean Neutrosophic α -closed set in Y for every Pythagorean Neutrosophic closed set A in X.
3- A Pythagorean Neutrosophic continuous function if $f^{-1}(B)$ is a Pythagorean Neutrosophic open set in X for every Pythagorean Neutrosophic open set B in Y.
4- A Pythagorean Neutrosophic α- continuous function if $f^{-1}(B)$ is a Pythagorean Neutrosophic α-open set in X for every Pythagorean Neutrosophic open set B in Y.

Definition 2.13: A mapping $f: (X, T) \to (Y, S)$ from a Pythagorean Neutrosophic topological space (X, T) to another Pythagorean Neutrosophic topological space (Y, S) is said to be

1- A Pythagorean Neutrosophic α^***-**continuous function if $f^{-1}(B)$ is a Pythagorean Neutrosophic α-open set in X for every Pythagorean Neutrosophic α-open set B in Y.
3- A Pythagorean Neutrosophic α^{**}-continuous function if $f^{-1}(B)$ is a Pythagorean Neutrosophic open set in X for every Pythagorean Neutrosophic α-open set B in Y.

Definition 2.14: A PNTS $(X, \tau\ )$ is called a Pythagorean neutrosophic

i. T_0-space or (PN T_0- space, for short) if for every pair of Pythagorean Neutrosophic points $p = x_{(\mu,\sigma,\gamma)}, q = y_{(\alpha,\beta,\theta)}$ with $x \neq y$, ther exists a Pythagorean Neutrosophic α -open sets A and B such that $(p \subseteq A, q \nsubseteq A)$ and $q \subseteq B, p \nsubseteq B$.
ii. T_1-space (PN T_1-space, for short) iff for every pair of Pythagorean Neutrosophic points $p = x_{(\mu,\sigma,\gamma)}, q = y_{(\alpha,\beta,\theta)}$ with $x \neq y$, ther exists a Pythagorean Neutrosophic open sets A and B such that $(p \subseteq A, q \nsubseteq A)$ and $q \subseteq B, p \nsubseteq B)$.
iii. T_2-space or neutrosophic Hausdorff space (PN T_2-space or Hausdorff space, for short) iff for any two Pythagorean Neutrosophic points $p = x_{(\mu,\sigma,\gamma)}, q = y_{(\alpha,\beta,\theta)}$ with $x \neq y$, there exist two Pythagorean Neutrosophic open sets A, B in X such that $x_{(\mu,\sigma,\gamma)} \in A, y_{(\alpha,\beta,\theta)} \in B$ and $A \cap B = 0$.

2. α-Separation Axioms in Pythagorean Neutrosophic Topologial Spaces:

Definition 3.1: A Pythagorean Neutrosophic Topological Spaces (X, τ) is said to be Pythagorean Neutrosophic Fuzzy α-T_0 (PNF α -T_0) if for every pair of Pythagorean Neutrosophic Fuzzy points $p = x_{(\mu,\sigma,\gamma)}, q = y_{(\alpha,\beta,\theta)}$ with different supports, there exists a Pythagorean Neutrosophic Fuzzy α -open set A such that either. $(p \subseteq A, q \nsubseteq A)$ or $(q \subseteq A, p \nsubseteq A)$.

Theorem 3.2: A Pythagorean Neutrosophic Topological Spaces (X, τ) is PN α -T_0 if and only if any two crisp Pythagorean Neutrosophic Fuzzy points with different supports have disjoint Pythagorean Neutrosophic Fuzzy α -closure.

Proof:

Let (X, τ) be A Pythagorean Neutrosophic Fuzzy α -T_0 and $p = x_{(\mu,\sigma,\gamma)}, q = y_{(\alpha,\beta,\theta)}$ be two crisp Pythagorean Neutrosophic Fuzzy points with supports x, y respectively, where $x \neq y$. Since (X, τ) is PN α -T_0 there exists a Pythagorean Neutrosophic α -open set A such that either $(p \subseteq A, q \nsubseteq A)$ or $q \subseteq A, p \nsubseteq A)$.

If $p \subseteq A, q \nsubseteq A$, this implies that $q \subseteq PN\alpha cl(q), PN\alpha(q) \nsubseteq A$, since $p \nsubseteq A^c, p \nsubseteq (PN\alpha cl(q))^c$. But $p \subseteq PN\alpha cl(p)$. There fore $PN\alpha cl(p) \neq PN\alpha cl(q)$.

Conversely, let p and q be any two Pythagorean Neutrosophic Fuzzy points with different supports x, y respectively. Let p_1, q_1 be Pythagorean Neutrosophic Fuzzy points such that $p_1(x) = q_1(y) = 1$. By hypothesis $PN\alpha cl(p_1) \neq PN\alpha cl(q_1)$, but $p \leq p_1$ implies that $p^c \geq p_1^c \geq (PN\alpha cl(p_1))^c$. Thus $(PN\alpha cl(p_1))^c$ is A Pythagorean Neutrosophic α -open set such that $q \nsubseteq (PN\alpha cl(p_1), p \subseteq PN\alpha cl(p_1)$. Hence (X, τ) is Pythagorean Neutrosophic α -T_0.

Theorem 3.3: If $f:(X,T) \to (Y,S)$ is an injective, Pythagorean Neutrosophic α -continuous mapping and (Y,S) is a Pythagorean Neutrosophic α -T_0 Space, then (X,T) is Pythagorean Neutrosophic α -T_0 Space.

Proof:

Let $p = x_{(\mu,\sigma,\gamma)}, q = y_{(\alpha,\beta,\theta)}$ be a Pythagorean Neutrosophic points with different supports in X. Then $f(p), f(q)$ are two Pythagorean Neutrosophic points with different supports in Y. Since (Y,S) is a Pythagorean Neutrosophic $\alpha - T_0$Space, then there exists a Pythagorean Neutrosophic open set A such that $f(p) \subseteq A, f(q) \nsubseteq A$ or $f(q) \subseteq A, f(p) \nsubseteq A$. Consider the part $f(q) \subseteq A, f(p) \nsubseteq A$. It follows that $q \subseteq f^{-1}(A), P \nsubseteq f^{-1}(A)$. Where $f^{-1}(A)$ is a Pythagorean Neutrosophic α -open set in X. Hence (X,T) is a Pythagorean Neutrosophic α -T_0 Space.

Theorem 3.4: Let f be continuous mapping from a Pythagorean Neutrosophic Topological Space (X,T) into Pythagorean Neutrosophic Topological Space (Y,S). If (Y,S) is a Pythagorean Neutrosophic Fuzzy α-T_0 Space, then so is (X,T).

Proof:

Let $p = x_{(\mu,\sigma,\gamma)}, q = y_{(\alpha,\beta,\theta)}$ are Pythagorean Neutrosophic Fuzzy points with different supports in X, then $f(p)$ and $f(q)$ are two Pythagorean Neutrosophic points with different supports in Y. Since (Y,S) is Pythagorean Neutrosophic α -T_0 space, then there exists a Pythagorean Neutrosophic open set A such that $f(p) \subseteq A, f(q) \nsubseteq A$, or $f(q) \subseteq A, f(p) \nsubseteq A$. Consider the part. If $f(p) \subseteq A, f(q) \nsubseteq A$. Its follows that, $p \subseteq f^{-1}(A), q \nsubseteq f^{-1}(A)$, where $f^{-1}(A)$ is a Pythagorean Neutrosophic α -open set in X. Hence (X,T) is a Pythagorean Neutrosophic α -T_0 Space.

Theorem 3.5: Let f be an injective Pythagorean Neutrosophic fuzzy α^* continuous mapping from a Pythagorean Neutrosophic topological space (X,τ) into Pythagorean Neutrosophic topological space (Y,S). If (Y,S) is a Pythagorean Neutrosophic $\alpha - T_0$ space, then so is (X,τ).

Proof:

Let $p = x_{(\mu,\sigma,\gamma)}, q = y_{(\alpha,\beta,\theta)}$ are Pythagorean Neutrosophic points with different supports in X, then $f(p)$ and $f(q)$ are two Pythagorean Neutrosophic points with different supports in Y. Since (Y,S) is Pythagorean Neutrosophic α -T_0 space, then there exists a Pythagorean Neutrosophic $\alpha -$ open set A such that $f(p) \subseteq A, f(q) \nsubseteq A$, or $f(q) \subseteq A, f(p) \nsubseteq A$. Consider the part. If $f(p) \subseteq A, f(q) \nsubseteq A$. Its follows that, $p \subseteq f^{-1}(A), q \nsubseteq f^{-1}(A)$, where $f^{-1}(A)$ is a Pythagorean Neutrosophic α -open set in X. Hence (X,T) is a Pythagorean Neutrosophic α -T_0 Space.

Theorem 3.6: If $f:(X,\tau) \to (Y,S)$ is an injective, Pythagorean Neutrosophic α^{**}continuous mapping and (Y,S) is a Pythagorean Neutrosophic $\alpha - T_0$ space, then (X,τ) is a Pythagorean Neutrosophic $\alpha - T_0$ space.

Proof:

Let $p = x_{(\mu,\sigma,\gamma)}, q = y_{(\alpha,\beta,\theta)}$ are Pythagorean Neutrosophic points with different supports in X, then $f(p)$ and $f(q)$ are two Pythagorean Neutrosophic points with different supports in Y. Since (Y,S) is Pythagorean Neutrosophic α -T_0 space, then there exists a Pythagorean Neutrosophic $\alpha -$ open set A such that $f(p) \subseteq A, f(q) \nsubseteq A$, or $f(q) \subseteq A, f(p) \nsubseteq A$. Consider the part. If $f(p) \subseteq A, f(q) \nsubseteq A$. Its follows that, $p \subseteq f^{-1}(A), q \nsubseteq f^{-1}(A)$, where $f^{-1}(A)$ is a Pythagorean Neutrosophic open set in X. Hence (X,T) is a Pythagorean Neutrosophic α -T_0 Space.

Definition 3.7: A Pythagorean Neutrosophic Topological Space (X,τ) is said to be Pythagorean Neutrosophic α -T_1 (PNα-T_1) if for every pair of Pythagorean Neutrosophic points $p = x_{(\mu,\sigma,\gamma)}, q = y_{(\alpha,\beta,\theta)}$ with $x \neq y$, ther exists a Pythagorean Neutrosophic α -open sets A and B such that $(p \subseteq A, q \nsubseteq A)$ and $q \subseteq B, p \nsubseteq B)$.

Theorem 3.8: A Pythagorean Neutrosophic Topological Spaces (X,τ) is Pythagorean Neutrosophic α -T_1 if and only if every crisp Pythagorean Neutrosophic point is a Pythagorean Neutrosophic α -closed set.

Proof:

Let $p = x_{(1,0,0)}$ be a PNCP in X. Also let $q = y_{(\alpha,\beta,\theta)} \in p^c$ be any PNP. Then obviously $x \neq y$. Since X is a PN α -T_1, so for $y_{(\alpha,\beta,\theta)}$ and $x_{(\mu,\sigma,\gamma)}$, there exists a PN α*open* set A such that $y_{(\alpha,\beta,\theta)} \in A$ and $x_{(\mu,\sigma,\gamma)} \notin A$. Since for all μ, σ, γ with $0 > \mu \leq 1, 0 \leq \sigma < 1, 0 \leq \gamma < 1$, one such A exists, therefore we must have a PNα open set B such that $y_{(\alpha,\beta,\theta)} \in B$ and $p \cap B = \emptyset$. Thereforeq $\in B \subseteq p^c$. So, p^c is a PNα open set and consequently, p is a PNα closed set.

Conversely, let $p_1 = x_{1(\mu_1,\sigma_1,\gamma_1)}$ and $p_2 = x_{2(\mu_2,\sigma_2,\gamma_2)}$ are Pythagorean Neutrosophic Fuzzy point with different supports x_1 and x_2.Also let $q_1 = x_{1(\alpha_1,\beta_1,\theta_1)}$ and $q_2 = x_{2(\alpha_2,\beta_2,\theta_2)}$ be a Pythagorean Neutrosophic Fuzzy point with different supports x_1 and x_2, respectively and such that $q_1(x_1) = q_2(x_2) = 1$. The Pythagorean Neutrosophic Fuzzy set q_1^c *and* q_2^c are Pythagorean Neutrosophic Fuzzy α -open and satisfy the conditions $p_1 \subseteq q_2^c, p_2 \nsubseteq q_2^c$ and $p_2 \subseteq q_1^c, p_1 \nsubseteq q_2^c$. Hence the space (X, τ) is PN α -T_1.

Theorem 3.9: Every Pythagorean Neutrosophic α -T_1 Space is obviously be Pythagorean Neutrosophic α -T_0 Space. .

Proof:

Let $p = x_{(\mu,\sigma,\gamma)}, q = y_{(\alpha,\beta,\theta)}$ with $x \neq y$, be two PNS in X. Since X is PNα -T_1 space, so there exists Pythagorean Neutrosophic α -open sets A and B such that $(p \subseteq A, q \nsubseteq A)$ and $(q \subseteq B, p \nsubseteq B)$. Hence (X, τ) is PNα -T_0 space.

Remark 3.10: Converse of the above theorem is not true in general. We establish it by the following counter example.

Example 3.11: Let $X = \{a, b\}$ and $\tau = \{0,1, A\}$, where $A = \{\langle a, 1,0,0\rangle, \langle b, 0,1,1\rangle\}$. Clearly the (X, τ) is a PN α -T_0 space. But (X, τ) is not an PN α -T_1 space, because the PNP $a_{(1,0,0)}$ is not PNαC set. Therefore, by the theorem (3.6) is not an PNα-T_1 space. Thus PN(X, τ) is a PN α -T_0 space but not PNα-T_1 space.

Theorem 3.12: Let f be a bijective Pythagorean Neutrosophic α -open function from a PNTS (X, τ) to another PNTS (Y, σ). If (X, τ) is PN-T_1 then (Y, σ) is an PNα -T_1 space.

Proof:

Let $y_{1(\mu,\sigma,\gamma)}, y_{2(\mu',\sigma',\gamma')}$ with $y_1 \neq y_2$, be two PNPs in Y. Since f is bijective, so there exist two PNPs $x_{1(\alpha,\beta,\theta)}, x_{2(\alpha',\beta',\theta')}, x_1 \neq x_2$ in X such that $f\left(x_{1(\alpha,\beta,\theta)}\right) = y_{1(\mu,\sigma,\gamma)}, f\left(x_{2(\alpha',\beta',\theta')}\right) = y_{2(\mu',\sigma',\gamma')}$. Since X is PN-T_1, so there exists open set A such that $x_{1(\alpha,\beta,\theta)} \in A, x_{2(\alpha',\beta',\theta')} \notin A$ and there exists open set B such that $x_{2(\alpha',\beta',\theta')} \in B, x_{1(\alpha,\beta,\theta)} \notin B$. Since f is a Pythagorean Neutrosophic α-open function, so $f(A)$ is a PNα open set such that $f\left(x_{1(\alpha,\beta,\theta)}\right) = y_{1(\mu,\sigma,\gamma)} \in f(A)$ and $f\left(x_{2(\alpha',\beta',\theta')}\right) = y_{2(\mu',\sigma',\gamma')} \notin f(A)$. Similarly $f(B)$ is a PN α open set such that $f\left(x_{1(\alpha,\beta,\theta)}\right) = y_{1(\mu,\sigma,\gamma)} \notin f(B)$ and $f\left(x_{2(\alpha',\beta',\theta')}\right) = y_{2(\mu',\sigma',\gamma')} \in f(B)$. Thus for any two PNPs $y_{1(\mu,\sigma,\gamma)}, y_{2(\mu',\sigma',\gamma')}$ in Y such that $y_1 \neq y_2$, there exists a PN α open set $f(A)$ such that $y_{1(\mu,\sigma,\gamma)} \in f(A), y_{2(\mu',\sigma',\gamma')} \notin f(A)$. And there exists a PN α -open set $f(B)$ such that $y_{2(\mu',\sigma',\gamma')} \in f(B), y_{1(\mu,\sigma,\gamma)} \notin f(B)$. Therefore, (Y, σ) is PN α -T_1 space.

Theorem 3.13: Let f be a injective Pythagorean Neutosophic α -continuous function from a PNTS(X, τ) to another PNTS(Y, σ). If (Y, σ) is PN T_1 then (X, τ) is an PNα -T_1 space.

Proof:

Let $x_{1(\alpha,\beta,\theta)}, x_{2(\alpha',\beta',\theta')}$, be any two PNPs in X such that $x_1 \neq x_2$. Since f is one to one, so there exist two PNPs $y_{1(\mu,\sigma,\gamma)}, y_{2(\mu',\sigma',\gamma')}$ with $y_1 \neq y_2$, in Y such that $f\left(x_{1(\alpha,\beta,\theta)}\right) = y_{1(\mu,\sigma,\gamma)}$ and $f\left(x_{2(\alpha',\beta',\theta')}\right) = y_{2(\mu',\sigma',\gamma')}$,i.e.

$x_{1(\alpha,\beta,\theta)} = f^{-1}\left(y_{1(\mu,\sigma,\gamma)}\right)$ *and* $x_{2(\alpha',\beta',\theta')} = f^{-1}\left(y_{2(\mu',\sigma',\gamma')}\right)$. Since Y is PN T_1, so there exists Pythagorean Neutrosophic open set A such that $y_{1(\mu\sigma\gamma)} \in A, y_{2(\mu',\sigma',\gamma')} \notin A$ and there exists Pythagorean Neutrosophic open set B such that $y_{2(\mu',\sigma',\gamma')} \in B, y_{1(\mu,\sigma,\gamma)} \notin B$. Since f is a Pythagorean Neutrosophic α -continuous function, so $f^{-1}(A)$ and $f^{-1}(B)$ are Pythagorean Neutrosophic α open sets in X. Also $y_{1(\mu,\sigma,\gamma)} \in A \Rightarrow f^{-1}(y_{1(\mu,\sigma,\gamma)}) \in f^{-1}(A) \Rightarrow x_{1(\alpha,\beta,\theta)} \in f^{-1}(A)$ and $y_{2(\mu',\sigma',\gamma')} \notin A \Rightarrow f^{-1}\left(y_{2(\mu',\sigma',\gamma')}\right) \notin f^{-1}(A) \Rightarrow x_{2(\alpha',\beta',\theta')} \notin f^{-1}(A)$. Similarly $x_{2(\alpha',\beta',\theta')} \in f^{-1}(B)$ and $x_{1(\alpha,\beta,\theta)} \notin f^{-1}(B)$. Thus for any two Pythagorean Neutrosophic points $x_{1(\alpha,\beta,\theta)}, x_{2(\alpha',\beta',\theta')}$ in X such that $x_1 \neq x_2$, there exists a Pythagorean Neutrosophic α -open set $f^{-1}(A)$ such that $x_{1(\alpha,\beta,\theta)} \in f^{-1}(A), x_{2(\alpha',\beta',\theta')} \notin f^{-1}(A)$ and there exists a Pythagorean Neutrosophic α -open set $f^{-1}(B)$. Such that $x_{2(\alpha',\beta',\theta')} \in f^{-1}(B)$ and $x_{1(\alpha,\beta,\theta)} \notin f^{-1}(B)$. Therefore (X, τ) is an PN α -T_1 space.

Theorem 3.14: Let f be a injective Pythagorean neutrosophic α^*-continuous function from a PNTS(X, τ) to another PNTS(Y, σ). If (Y, σ) is PN α -T_1 then so is X

Proof:

Let $x_{1(\alpha,\beta,\theta)}, x_{2(\alpha',\beta',\theta')}$, be any two PNPs in X such that $x_1 \neq x_2$. Since f is one to one, so there exist two PNPs $y_{1(\mu,\sigma,\gamma)}, y_{2(\mu',\sigma',\gamma')}$ with $y_1 \neq y_2$, in Y such that $f\left(x_{1(\alpha,\beta,\theta)}\right) = y_{1(\mu,\sigma,\gamma)}$ and $f\left(x_{2(\alpha',\beta',\theta')}\right) = y_{2(\mu',\sigma',\gamma')}$, i.e. $x_{1(\alpha,\beta,\theta)} = f^{-1}\left(y_{1(\mu,\sigma,\gamma)}\right)$ and $x_{2(\alpha',\beta',\theta')} = f^{-1}\left(y_{2(\mu',\sigma',\gamma')}\right)$. Since Y is PNα T_1, so there exists Pythagorean Neutrosophic α -open set A such that $y_{1(\mu,\sigma,\gamma)} \in A, y_{2(\mu',\sigma',\gamma')} \notin A$ and there exists Pythagorean Neutrosophic α -open set B such that $y_{2(\mu',\sigma',\gamma')} \in B, y_{1(\mu,\sigma,\gamma)} \notin B$. Since f is a Pythagorean Neutrosophic α^*-continuous function, so $f^{-1}(A)$ and $f^{-1}(B)$

***Received**: 18-02-2026 - **Accepted**: 15-04-2026 - **Published**: 22-04-2026*

are Pythagorean Neutrosophic α -open sets in X. Also $y_{1(\mu,\sigma,\gamma)} \in A \Longrightarrow f^{-1}\left(y_{1(\mu,\sigma,\gamma)}\right) \in f^{-1}(A) \Longrightarrow x_{1(\alpha,\beta,\theta)} \in f^{-1}(A)$ and $y_{2(\mu,\sigma,\gamma)} \notin A \Longrightarrow f^{-1}\left(y_{2(\mu',\sigma',\gamma')}\right) \notin f^{-1}(A) \Longrightarrow x_{2(\alpha',\beta',\theta')} \notin f^{-1}(A)$. Similarly $x_{2(\alpha',\beta',\theta')} \in f^{-1}(B)$ and $x_{1(\alpha,\beta,\theta)} \notin f^{-1}(B)$. Thus for any two Pythagorean Neutrosophic points $x_{1(\alpha,\beta,\theta)}, x_{2(\alpha',\beta',\theta')}$ in X such that $x_1 \neq x_2$, there exists a Pythagorean Neutrosophic α -open set $f^{-1}(A)$ such that $x_{1(\alpha,\beta,\theta)} \in f^{-1}(A), x_{2(\alpha',\beta',\theta')} \notin f^{-1}(A)$ and there exists a Pythagorean Neutrosophic α -open set $f^{-1}(B)$. Such that $x_{2(\alpha',\beta',\theta')} \in f^{-1}(B)$ and $x_{1(\alpha,\beta,\theta)} \notin f^{-1}(B)$. Therefore, (X,τ) is an PN α -T_1 space.

Theorem 3.15: Let f be a injective Pythagorean neutrosophic α^{**}-continuous function from a PNTS(X,τ) to another PNTS(Y,σ). If (Y,σ) is PN α -T_1 then so is X

Proof:

Similar to that of theorem (3.6).

Definition 3.16: A Pythagorean Neutrosophic topological space (X, T) is said to be a Pythagorean Neutrosophic stronger space$-\alpha T_1$ if every Pythagorean Neutrosophic point is a Pythagorean Neutrosophic α-closed set

Definition 3.17: A Pythagorean Neutrosophic Topological Space (X,τ) is said to be Pythagorean Neutrosophic α -T_2 space or Pythagorean Neutrosophic α -Hausdorff space (PNαT_2 -space or PN α -Hausdorff space) iff for any two Pythagorean Neutrosophic points $p = x_{(\mu,\sigma,\gamma)}, q = y_{(\alpha,\beta,\theta)}$ with $x \neq y$, there exist two Pythagorean Neutrosophic α -open sets A, B in X such that $x_{(\mu,\sigma,\gamma)} \in A, y_{(\alpha,\beta,\theta)} \in B$ and $A \cap B = 0$.

Theorem 3.18: Let (X,τ) be a PNTS. If X is PNT_2 -space then X is a PN αT_2-space.

Proof:

Let $x_{(\mu,\sigma,\gamma)}, y_{(\alpha,\beta,\theta)}$ with $x \neq y$, be two PNPs in X. Since (X,τ) is a PN T_2-space, so there exist Pythagorean Neutrosophic open sets A, B such that $x_{(\mu,\sigma,\gamma)} \in A, y_{(\alpha,\beta,\theta)} \in B$ and $A \cap B = 0$. Since every Pythagorean Neutrosophic open set is a Pythagorean Neutrosophic α -open set, so for any two PNPs $x_{(\mu,\sigma,\gamma)}, y_{(\alpha,\beta,\theta)}$ with $x \neq y$, there exist Pythagorean Neutrosophic α -open sets A, B such that $x_{(\mu,\sigma,\gamma)} \in A, y_{(\alpha,\beta,\theta)} \in B$ and $A \cap B = 0$. Hence (X,τ) is a PN αT_2-space.

Remark 3.19: Converse of the theorem is not true in general. We establish it by the following counter example.

Example 3.20: Let $X = \{x, y\}$ and $\tau = \{0,1\}$ clearly (X,τ) is not PN T_2-space. We now show that (X,τ) is a PN αT_2-space. Let $x_{(\mu,\sigma,\gamma)} \in A, y_{(\alpha,\beta,\theta)} \in B$ be any two PNPs in X. Also let $A = \{\langle x,1,0,0\rangle, \langle y,0,1,1\rangle\}$ and $B = \{\langle x,0,1,1\rangle, \langle y,1,0,0\rangle\}$. Clearly, A and B are two Pythagorean Neutrosophic α -open sets in X such that $A \cap B = 0$. Thus there exist Pythagorean Neutrosophic α -open sets A and B, $x_{(\mu,\sigma,\gamma)} \in A, y_{(\alpha,\beta,\theta)} \in B$ and $A \cap B = 0$. Therefore, (X,τ) is a PN $\alpha{-}T_2$space.

Theorem 3.21: Let (X,τ) be a PN $\alpha{-}T_2$ space. Then every Pythagorean Neutrosophic subspace of X is a PN$\alpha - T_2$ space, hence the property is hereditary.

Proof:

Let $(Y, \tau|_Y)$ be a Pythagorean Neutrosophic subspace of X, where $\tau_Y = \left\{G_Y = \{\langle x, \mu_{G|_Y}, \sigma_{G|_Y}, \gamma_{G|_Y}, x \in Y, G \in \tau\rangle\}\right\}$ and $G = \langle x, \mu_G(x), \sigma_G(x), \gamma_G(x)\rangle$. Let $p = x_{(\mu,\sigma,\gamma)}, q = y_{(\alpha,\beta,\theta)}$ be two distinct PNPs in Y, that is they have distinct supports. Then, clearly $p = x_{(\mu,\sigma,\gamma)}, q = y_{(\alpha,\beta,\theta)}$ are also distinct PNPs in X and as X is PN $\alpha - T_2$, therefore there exist two Pythagorean Neutrosophic α -open sets A and B such that $x_{(\mu,\sigma,\gamma)} \in A, y_{(\alpha,\beta,\theta)} \in B$ and $A \cap B = 0$. Thus, there exist $A_Y, B_Y \in \tau_Y$ such that $x_{(\mu,\sigma,\gamma)} \in A_Y, y_{(\alpha,\beta,\theta)} \in B_Y$ and $A_Y \cap B_Y = 0$. hence the property is hereditary.

Theorem 3.22: Let (X,τ) be a PNTS. If (X,τ) is an PN $\alpha{-}T_2$space then it is a PN $\alpha - T_1$space. **Proof:**

Let $x_{(\mu,\sigma,\gamma)}, y_{(\alpha,\beta,\theta)}$ with $x \neq y$. Since (X,τ) is a PN $\alpha - T_2$space, so there exist Pythagorean Neutrosophic α -open sets A and B such that $x_{(\mu,\sigma,\gamma)} \in A, y_{(\alpha,\beta,\theta)} \in B$ and $A \cap B = 0$. Since $x_{(\mu,\sigma,\gamma)} \in A$ and $A \cap B = 0$, so $x_{(\mu,\sigma,\gamma)} \notin B$. Similarly $y_{(\alpha,\beta,\theta)} \notin A$. Thus for any two PNPs $x_{(\mu,\sigma,\gamma)}, y_{(\alpha,\beta,\theta)}$ with $x \neq y$ there exists a PN α open set A such that $x_{(\mu,\sigma,\gamma)} \in A, y_{(\alpha,\beta,\theta)} \notin A$ and there exists a PN α open set B such that $x_{(\mu,\sigma,\gamma)} \notin B, y_{(\alpha,\beta,\theta)} \in B$. Hence (X,τ) is a PN $\alpha - T_1$space.

Theorem 3.23: Let f be a bijective Pythagorean Neutrosophic α -open function from a PNTS(X,τ) to another PNTS(Y,σ). If (X,τ) is PN T_2 then (Y,σ) is an PN $\alpha - T_2$space.

Proof:

Let $y_{1(\mu,\sigma,\gamma)}$, $y_{2(\mu',\sigma',\gamma')}$ with $y_1 \neq y_2$, be two PNPs in Y. Since f is bijective, so there exist two PNPs $x_{1(\alpha,\beta,\theta)}$, $x_{2(\alpha',\beta',\theta')}$, $x_1 \neq x_2$ in X such that $f\left(x_{1(\alpha,\beta,\theta)}\right) = y_{1(\mu,\sigma,\gamma)}$, $f\left(x_{2(\alpha',\beta',\theta')}\right) = y_{2(\mu',\sigma',\gamma')}$. Since X is PN T_2, so there exists open set A and B such that $x_{1(\alpha,\beta,\theta)} \in A$, $x_{2(\alpha',\beta',\theta')} \in B$ and $A \cap B = 0$. Since f is a Pythagorean Neutrosophic α -open function, so $f(A), f(B)$ are PN α open sets such that $f\left(x_{1(\alpha,\beta,\theta)}\right) = y_{1(\mu,\sigma,\gamma)} \in f(A)\ and\ x_{2(\alpha',\beta',\theta')} = f\left(y_{2(\mu',\sigma',\gamma')}\right) \in f(B)$. Since f is bijective, so $f(A) \cap f(B) = f(A \cap B) = f(0) = 0$.

Thus for any two PNs $y_{1(\mu,\sigma,\gamma)}, y_{2(\mu',\sigma',\gamma')}$ in Y such $y_1 \neq y_2$, there exists a PN α open set $f(A), f(B)$ such that $y_{1(\mu,\sigma,\gamma)} \in f(A)$, $y_{2(\mu',\sigma',\gamma')} \in f(B)$ and so $f(A) \cap f(B) = 0$. Therefore, (Y,σ) is an PN $\alpha - T_2$ space.

Theorem 3.24: If $f: (X,T) \to (Y,S)$ is an injective Pythagorean Neutrosophic α -continuous mapping and (Y,S) is a Pythagorean Neutrosophic T_2-space, then (X,T) is Pythagorean Neutrosophic $\alpha{-}T_2$ space.

Proof:

Similar to that of theorem (3.13).

Theorem 3.25: Let f be a one to one Pythagorean Neutrosophic α^* continuous function from a PNTS (X,τ) to another PNTS (Y,σ). If (Y,σ) is PN $\alpha - T_2$ then (X,τ) is an PN $\alpha - T_2$ space.

Proof:

Let $x_{1(\alpha,\beta,\theta)}, x_{2(\alpha',\beta',\theta')}$, be any two PNPs in X such that $x_1 \neq x_2$. Since f is one to one, so there exist two PNPs $y_{1(\mu,\sigma,\gamma)}, y_{2(\mu',\sigma',\gamma')}$ with $y_1 \neq y_2$, in Y such that $f\left(x_{1(\alpha,\beta,\theta)}\right) = y_{1(\mu,\sigma,\gamma)}, f\left(x_{2(\alpha',\beta',\theta')}\right) = y_{2(\mu',\sigma',\gamma')}$, i.e. $x_{1(\alpha,\beta,\theta)} = f^{-1}(y_{1(\mu,\sigma,\gamma)})$, $x_{2(\alpha',\beta',\theta')} = f^{-1}(y_{2(\mu',\sigma',\gamma')})$. Since Y is PN $-T_2$, so there exists Pythagorean Neutrosophic α open set A and B such that $y_{1(\mu,\sigma,\gamma)} \in A, y_{2(\mu',\sigma',\gamma')} \in B$ and $A \cap B = 0$. Since f is a Pythagorean Neutrosophic α^*-continuous function, so $f^{-1}(A)$ and $f^{-1}(B)$ are Pythagorean Neutrosophic α open sets in X. Also $y_{1(\mu,\sigma,\gamma)} \in A \Rightarrow f^{-1}\left(y_{1(\mu,\sigma,\gamma)}\right) \in f^{-1}(A) \Rightarrow x_{1(\alpha,\beta,\theta)} \in f^{-1}(A)$. Similarly $x_{2(\alpha',\beta',\theta')} \in f^{-1}(B)$. Thus for any two Pythagorean Neutrosophic points $x_{1(\alpha,\beta,\theta)}, x_{2(\alpha',\beta',\theta')}$ in X such that $x_1 = x_2$, there exists a Pythagorean Neutrosophic α open set $f^{-1}(A)$ and $f^{-1}(B)$. Such that $x_{1(\alpha,\beta,\theta)} \in f^{-1}(A)$, and $x_{2(\alpha',\beta',\theta')} \in f^{-1}(B), f^{-1}(A) \cap f^{-1}(B) = 0$. Therefore, (X,τ) is an PN $\alpha - T_2$ space.

Theorem 3.25: If $f: (X,\tau) \to (Y,\delta)$ is an injective, Pythagorean Neutrosophic $\alpha^{**} -$ continuous mapping and (Y,δ) is Pythagorean Neutrosophic $\alpha - T_2$ space, then $(X,\tau)$is Pythagorean Neutrosophic $T2$ -space.

Proof: Similar to that of theorem 3.6.

Definition 3.26: A Pythagorean Neutrosophic Topological Spaces (X,τ) is said to be Pythagorean Neutrosophic α -regular if for a for a Pythagorean Neutrosophic α closed set A and $x_{(\mu,\sigma,\gamma)} \notin A$ there exists a Pythagorean Neutrosophic α -open sets U and V such that. $x_{(\mu,\sigma,\gamma)} \in U, A \subseteq V$ and $U \cap V = 0$. Moreover the space (X,τ) is called a Pythagorean Neutrosophic α -T_3 if it is both α -normal and satisfies α -T_1 separation axiom.

Definition 3.27: A Pythagorean Neurosophic Topological Spaces (X,τ) is said to be Pythagorean Neutrosophic α -normal if for every pair of disjoint Pythagorean Neutrosophic α -closed set A and B there exists a Pythagorean Neutrosophic α -open sets U and V such that $A \subseteq U, B \subseteq V$ and $U \cap V = 0$. The space (X,τ) is called a Pythagorean Neutrosophic α -T_4 if it is both α -normal and satisfies α -T_1 separation axiom.

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***Received**: 18-02-2026 - **Accepted**: 15-04-2026 - **Published**: 22-04-2026*

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Received*: 18-02-2026 -* ***Accepted****: 15-04-2026 -* ***Published****: 22-04-2026*